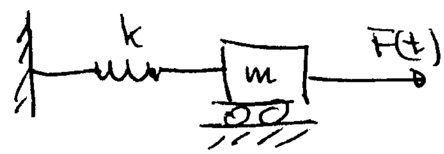
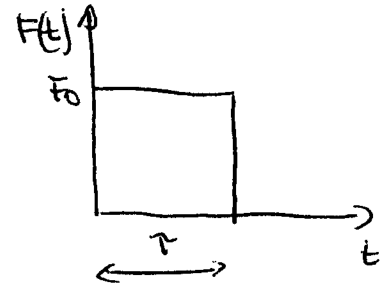
Transient Vibrations

Pulse and Impulse excitation

Pulse :





For $0 < t < \tau$.

$$m\ddot{x} + kx = F_0 \quad \Rightarrow$$

$$x = c_1 \sin\omega_n t + c_2 \cos\omega_n t + F_0/k$$

at $t = 0$ $\quad x(0) = \dot{x}(0) = 0$

so $\quad c_1 = 0 \qquad c_2 = -\dfrac{F_0}{k}$

$$x = \frac{F_0}{k}(1 - \cos\omega_n t) \quad \text{for } t < \tau$$

For $t > \tau$.

Let $\quad t' = t - \tau$

$$m\ddot{x} + kx = 0 \quad \Rightarrow \quad x = c_1' \sin\omega_n t' + c_2' \cos\omega_n t'$$

at $t = \tau$ $(t' = 0)$: $\quad \underset{\uparrow\, t'}{x(0)} = x(t = \tau)$, $\underset{\uparrow\, t'}{\dot{x}(0)} = \dot{x}(t = \tau)$

$$x = \frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t) + x(0)\cos\omega_n t$$

$$= \sqrt{\left(\frac{\dot{x}(0)}{\omega_n}\right)^2 + x(0)^2}\ \sin(\omega_n t' - \varphi)$$

but we know that

$$x(0) = \frac{F_0}{k}(1 - \cos\omega_n \tau)$$

$$\dot{x}(0) = \frac{F_0}{k}\omega_n \sin\omega_n \tau$$

$$x = \frac{F_0}{k}\sqrt{\sin^2\omega_n\tau + (1-\cos\omega_n\tau)^2}\;\sin(\omega t' - \psi)$$

$$x = \frac{2F_0}{k}\sqrt{\frac{1-\cos\omega_n\tau}{2}}\;\sin(\omega t' - \psi)$$

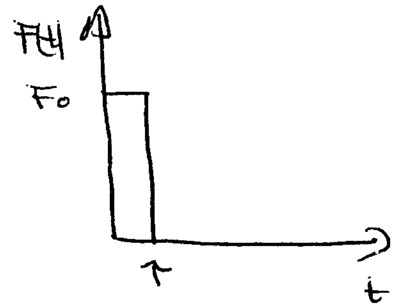
so amplitude of x is :

$$\underline{X} = \frac{2F_0}{k}\sqrt{\frac{1-\cos\omega_n\tau}{2}} \Rightarrow$$

$$\boxed{\frac{\underline{X}}{F_0/k} = 2\left|\sin\left(\frac{\omega_n\tau}{2}\right)\right|} \leftarrow \text{for pulse}$$

Impulse:

Consider a very short pulse
Assume the area $\hat{F}$ = constant
as $\tau \to 0$



For $t > \tau$ we know that

$$x(t'=0) = x(t=\tau) = -\frac{F_0}{k}\cos\omega_n\tau + \frac{F_0}{k}$$

but now let $\tau \to 0$

$$x(0) = -\frac{F_0}{k} + \frac{F_0}{k} = 0$$

Also

$$\dot{x}(0) = \dot{x}(t=\tau) = \frac{F_0}{k}\omega_n \sin\omega_n\tau$$

Let $\tau \to 0$

$\hat{F}$ = constant

$$\dot{x}(0) = \lim_{t\to 0}\frac{F_0}{k}\omega_n \sin\omega_n\tau = \lim_{\tau\to 0}\frac{F_0\cdot\tau}{\tau\cdot k}\omega_n(\omega_n\cdot\tau)$$

$$= \frac{\hat{F}}{k}\omega_n^2 = \frac{\hat{F}}{m}$$

since $c_1' = \dfrac{\dot{x}(0)}{\omega_n}$ and $c_2' = x(0)$

we get

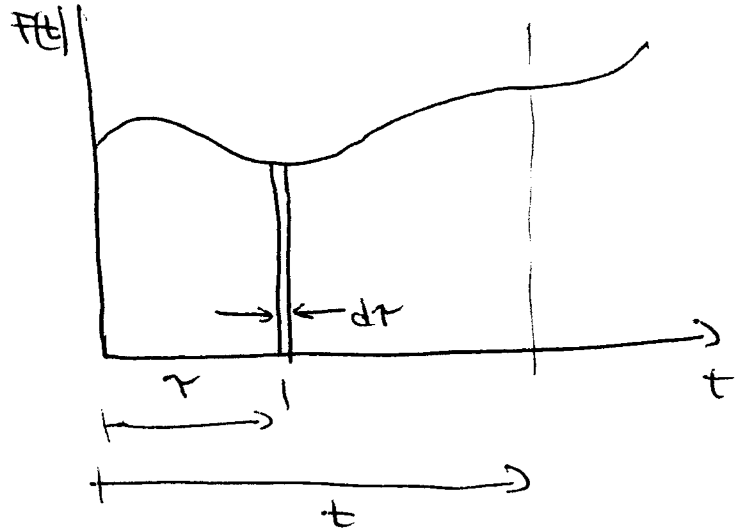
$$c_1' = \frac{\hat{F}}{m\omega_n} \quad , \quad c_2' = 0$$

$$\Rightarrow \boxed{x(t) = \frac{\hat{F}}{m\omega_n} \sin \omega_n t}$$

General excitation

Duhamels Integral:

$$m\ddot{x} + kx = F(t)$$



The system response to an impulse $\hat{F}$ is

$$\Delta x(t) = \frac{\hat{F}}{m\omega_n} \sin(\omega_n(t-\tau)) \qquad t \geq \tau$$

but $\hat{F} = F(\tau)\, d\tau \rightarrow$

$$\Delta x(t) = \frac{F(\tau)}{m\omega_n} \sin(\omega_n(t-\tau))\, d\tau \qquad t \geq \tau$$

Total response is obtained by superposition

$x(t) = \sum \Delta x(t)$ but as $d\tau \rightarrow 0$

$$\boxed{x(t) = \int dx = \int_0^t \frac{F(\tau)}{m\omega_n} \sin(\omega_n(t-\tau))\, d\tau}$$ Forced part

The total solution will be

$$x(t) = \frac{\dot{x}_0}{\omega_n} \sin\omega_n t + x_0 \cos\omega_n t + \frac{1}{m\omega_n} \int_0^t F(\tau) \sin\omega_n(t-\tau)\, d\tau$$

If damping is present:

$$x(t) = \left[\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin\omega_d t + x_0 \cos\omega_n t \right] e^{-\zeta\omega_n t} +$$

$$+ \frac{1}{m\omega_d} \int_0^t F(\tau) \cdot e^{-\zeta\omega_n (t-\tau)} \cdot \sin\omega_d (t-\tau)\, d\tau$$