

Normal Mode Approach (cont)

The modes are orthogonal w.r.t. M and K

Proof:

For the r^{th} mode we have

$$(M^{-1}K - \omega_r^2 I) \underline{\Delta}^r = 0$$

$$K \underline{\Delta}^r - \omega_r^2 M \underline{\Delta}^r = 0$$

Take the transpose of this equation

$$\underline{\Delta}^{rT} K - \omega_r^2 \underline{\Delta}^{rT} M = 0$$

Post multiply by $\underline{\Delta}^s$ (the s^{th} mode shape)

$$\underline{\Delta}^{rT} K \underline{\Delta}^s - \omega_r^2 \underline{\Delta}^{rT} M \underline{\Delta}^s = 0 \quad *$$

Pre multiply the equation

$$K \underline{\Delta}^s - \omega_s^2 M \underline{\Delta}^s = 0$$

by $\underline{\Delta}^{rT}$:

$$\underline{\Delta}^{rT} K \underline{\Delta}^s - \omega_s^2 \underline{\Delta}^{rT} M \underline{\Delta}^s = 0 \quad **$$

Subtract * from ** :

$$(\omega_r^2 - \omega_s^2) \underline{\Delta}^{rT} M \underline{\Delta}^s = 0$$

Hence we conclude that for distinct eigenvalues

$$\boxed{\underline{\Delta}^{rT} M \underline{\Delta}^s = 0 \quad r \neq s}$$

$\Rightarrow \underline{\Delta}^r$ and $\underline{\Delta}^s$ are orthogonal w.r.t M

Also $\boxed{\bar{X}^{rT} K \bar{X}^s = 0} \quad r \neq s$

$\Rightarrow \bar{X}^r$ and \bar{X}^s are also orthogonal w.r.t K .

Define the modal column matrix

$$A \equiv \left[\begin{array}{c} \bar{X}^1 \\ \bar{X}^2 \\ \vdots \\ \bar{X}^n \end{array} \right], \quad n \times n \text{ matrix}$$

It can be seen from $-\omega^2 M \bar{X} + K \bar{X} = 0$ that

$$-MA \begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots \\ & & & \omega_n^2 \end{bmatrix} + KA = 0$$

or

$$A^{-1} M^{-1} K A = \begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots \\ & & & \omega_n^2 \end{bmatrix}$$

Hence, the matrix A represents the coordinate transformation that diagonalizes the dynamical matrix H .

Also since

$$\underline{X}^{rT} M \underline{X}^s = 0 \quad r \neq s \quad \text{we get}$$

$$A^T M A = \begin{bmatrix} \diagdown & & \\ & 0 & \\ & & \diagdown \end{bmatrix} \rightarrow \underline{X}^{rT} M \underline{X}^r, \text{ non-zero diagonal}$$

and

$$A^T K A = \begin{bmatrix} \diagdown & & \\ & 0 & \\ & & \diagdown \end{bmatrix} \rightarrow \underline{X}^{rT} K \underline{X}^r$$

In previous example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Test orthogonality:

$$\begin{matrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} & \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} & = & \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & -4 \\ 1 & -3 & 4 \end{pmatrix} \\ \underline{A}^T & \underline{K} & \underline{A} & & & \end{matrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

The total solution will be

$$\underline{x}(t) = \sum_{i=1}^n \underline{\Delta}_i A_i \cos(\omega_i t + \phi_i)$$

Not A_i ; the modal column matrix.

The constants A_i and ϕ_i can be evaluated from the initial conditions

$$\underline{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} \quad \text{and} \quad \dot{\underline{x}} = \begin{pmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \vdots \\ \dot{x}_n(0) \end{pmatrix}$$

Principle Coordinate Approach

Let us seek a set of coordinates \underline{p} in which the system of DEQ's will be uncoupled. Consider the set of coordinates defined by

$$\underline{x} = T \underline{p} \quad \text{where } T \text{ is a coordinate transfer matrix.}$$

The DEQ becomes

$$M \ddot{\underline{x}} + K \underline{x} = 0$$

$$M T \ddot{\underline{p}} + K T \underline{p} = 0$$

$$\ddot{\underline{p}} + T^{-1} M^{-1} K T \underline{p} = 0$$

We showed previously that

$$A^{-1}(M^{-1}K)A = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix} \text{ (diagonal matrix)}$$

Hence the system will be uncoupled if we select T such that

$$T = A \cdot \underbrace{\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}}_{\text{constants}}$$

Then T satisfies the orthogonality condition

$$\boxed{T^T M T = \begin{bmatrix} a_i & \\ & \end{bmatrix}} \quad \boxed{T^T K T = \begin{bmatrix} b_i & \\ & \end{bmatrix}}$$

$$\text{so } T^T M T \ddot{\vec{p}} + T^T K T \vec{p} = 0$$

$$a_i \ddot{p}_i + b_i p_i = 0 \quad i = 1, \dots, n$$

Uncoupled eqns. of motion

These coordinates p_i are called principle coordinates.