Kinetic Energy

$$T = T(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$$

$$T = \frac{1}{2} \sum_{k=1}^{m} m_k V_k^2$$

where $V_k^2 = \dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2$

$$x_k = x_k(q_1, q_2, \ldots, q_n), \quad y_k = y_k(\quad), \quad z_k = z_k(\quad)$$

$$\frac{dx_k}{dt} = \dot{x}_k = \frac{\partial x_k}{\partial q_1}\dot{q}_1 + \frac{\partial x_k}{\partial q_2}\dot{q}_2 + \cdots$$

$$V_k^2 = \left[\left(\frac{\partial x_k}{\partial q_1}\right)^2 + \left(\frac{\partial y_k}{\partial q_1}\right)^2 + \left(\frac{\partial z_k}{\partial q_1}\right)^2\right]\dot{q}_1^2 + \left[\begin{matrix}\text{same but}\\ \text{wrt } q_2\end{matrix}\right]\dot{q}_2^2 + \cdots$$

$$+ \left[\left(\frac{\partial x_k}{\partial q_1}\right)\left(\frac{\partial x_k}{\partial q_2}\right) + \left(\frac{\partial y_k}{\partial q_1}\right)\left(\frac{\partial y_k}{\partial q_2}\right) + \left(\frac{\partial z_k}{\partial q_1}\right)\left(\frac{\partial z_k}{\partial q_2}\right)\right]\dot{q}_1\dot{q}_2 + \cdots$$

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j + O(\dot{q}^3)$$

where $m_{ij} = \sum_{k=1}^{m} m_k \left[\frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j} + \frac{\partial y_k}{\partial q_i}\frac{\partial y_k}{\partial q_j} + \frac{\partial z_k}{\partial q_i}\frac{\partial z_k}{\partial q_j}\right]$

Expanding m_{ij} about

$$m_{ij} = m_{ij}\Big|_0 + \sum_{\ell=1}^{n}\frac{\partial m_{ij}}{\partial q_\ell}\Big|_0 q_\ell + \frac{1}{2}\sum_{\ell=1}^{n}\sum_{p=1}^{n}\frac{\partial^2 m_{ij}}{\partial q_\ell \partial q_p}\Big| q_\ell q_p + \cdots$$

If we want to keep T to second order in q as we have for V we must neglect all terms in the expansion of higher order than $O(1)$

Multi Degree of Freedom

Potential Energy:

$$V = V(q_1, q_2, \dots, q_n)$$

Select the reference potential $V_0 = 0$ and assume that this represents the equilibrium configuration.

Then

$$V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i q_j$$

where $k_{ij} = \dfrac{\partial^2 V}{\partial q_i \partial q_j}$

Note that $k_{ij} = k_{ji}$ symmetric

If $k_{ij} \neq 0$ when $i \neq j$ then the system is said to be statically coupled.

$$\Rightarrow \quad m_{ij} = m_{ij}\big|_0 = \text{const}$$

Again note that $m_{ij} = m_{ji}$

so $T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_1 \dot{q}_2$

if $m_{ij} \neq 0$ when $i=j$ the system is said to be dynamically coupled.

Define

K = stiffness matrix $K = \{k_{ij}\}$

M = mass matrix $M = \{m_{ij}\}$

q - vector of elements $q_1, q_2 \cdots$

$$\underset{\sim}{q} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix}$$

We note that the potential energy can be expressed as

$$V = \frac{1}{2} \underset{\sim}{q}^T K \underset{\sim}{q} \geq 0 \quad \forall q$$

also

$$T = \frac{1}{2} \dot{\underset{\sim}{q}}^T M \underset{\sim}{q} \geq 0 \quad \forall \dot{q}$$

(Both K and M are positive definite)

Substitute into L.E.

$$\frac{d}{dt} \frac{\partial (T-V)}{\partial \underset{\sim}{q}_i} - \frac{\partial}{\partial q_i}(T-V) = 0$$

$$\Big\downarrow \text{ show}$$

$$\boxed{M \ddot{\underset{\sim}{q}} + K \underset{\sim}{q} = \underset{\sim}{0}}$$

Solution of $M\ddot{\underset{\sim}{x}} + K\underset{\sim}{x} = 0$

Normal Mode Approach

Let $\underset{\sim}{x} = \underline{X} \cdot e^{i\omega t}$

substitute into DEQ:

$$(-\omega^2 M\underline{X} + K\underline{X})e^{i\omega t} = \underset{\sim}{0}$$

$$(M^{-1}K - \omega^2 I)\underset{\sim}{\underline{X}} = 0 \qquad \text{(eigen value problem)}$$

For nontrivial solutions:

$$\det(M^{-1}K - \omega^2 I) = 0 \leftarrow \text{characteristic equation}$$

The characteristic equation will be an n^{th} order polynomial in ω^2 where n is ~~[illegible]~~ the system.

($M^{-1}K = H$: The Dynamical Matrix)

Assume that all of the roots of $\det(M^{-1}K - \omega^2 I)$ are distinct.

Then for each root (eigenvalue), ω_r, there will be a corresponding modal vector (eigenvector) $\underset{\sim}{\underline{X}}^r$

$$(M^{-1}K - \omega_r^2 I)\underline{X}^r = 0$$

solve for $\underline{X}^r$ to get mode shape.