Lagrange Multiplier

To find the constraint force we must allow a virtual displacement to have a component in the direction of the constraint force, i.e. introduce an additional generalized coordinate which allows the constraint in question to be violated.

Assume we have one constraint force $\underset{\sim}{f}_i^c$ acting on the i^{th} particle

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j + Q_j^c$$

where $Q_j^c = \underset{\sim}{f}_i^c \cdot \frac{\partial r_i}{\partial q_j}$

The associated constraint is given by

$$g(q_j,) = 0$$

Let

$$\underset{\sim}{f}_i^c \perp \text{ to } g = \text{const} \Rightarrow \underset{\sim}{f}_i^c = F(\underset{\sim}{\hat{\nabla}} g)$$

Laplacian

unit vector normal to the constraint force

magnitude of the constraint force

so

$$Q_j^c = \frac{F\left(\frac{\partial g}{\partial x}\underset{\sim}{i} + \frac{\partial g}{\partial y}\underset{\sim}{j} + \frac{\partial g}{\partial z}\underset{\sim}{k}\right)}{|\nabla g|} \cdot \underbrace{\left(\frac{\partial x_i}{\partial q_j}\underset{\sim}{i} + \frac{\partial y_i}{\partial q_j}\underset{\sim}{j} + \frac{\partial z_i}{\partial q_j}\underset{\sim}{k}\right)}_{\frac{\partial r_i}{\partial q_j}}$$

$$Q_j^c = \frac{F}{|\nabla g|}\left(\frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial g}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial g}{\partial z_i}\frac{\partial z_i}{\partial q_j}\right)$$

$$Q_j^c = \lambda_i \frac{\partial g}{\partial q_j}$$ chain rule

where $\lambda_i = \frac{F}{|\nabla g|}$

We would like to combine this with the $\frac{\partial}{\partial q_j}$ term on the LHS of the L.E.

Note that for holonomic constraint

$\frac{\partial g}{\partial \dot{q}} = 0$ (does not depend on velocity)

so we can write

$$\boxed{\frac{d}{dt}\frac{\partial(L+\lambda g)}{\partial \dot{q}_j} - \frac{\partial(L+\lambda g)}{\partial q_j} = Q_j \quad j = 1,\dots n}$$

we now have $n+1$ eqns for $n+2$ unknowns $(q_1, q_2 \dots, q_{n+1}, \lambda)$

Reimpose the constraint to eliminate q_{n+1}

solve for $q_1 \dots, q_n, \lambda$. Then $|f_i^c| = \lambda \cdot |\nabla g|$