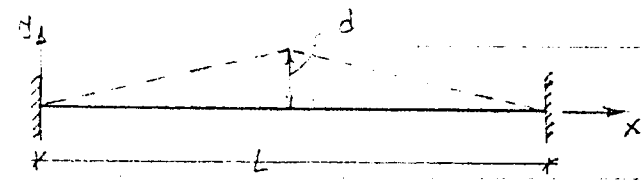
HW5 ME 588 solutions

(1)

the differential equation governing the vibration of a string stretched by a constant tension T and having a mass per unit length μ, is given by



$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

Assuming $y(x,t) = X(x) \cdot T(t)$

substituting in the differential equation, we get

$$X''(x) \cdot T(t) = \frac{\mu}{T} T''(t) \cdot X(x)$$

$$\therefore \frac{T}{\mu}\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \text{constant} = -\omega^2$$

$$\therefore T''(t) + \omega^2 T(t) = 0 \quad \Rightarrow T(t) = A\cos\omega t + B\sin\omega t$$

$$\therefore X''(x) + \frac{\mu}{T}\omega^2 X(x) = 0 \quad \Rightarrow X(x) = C\cos\omega\sqrt{\frac{\mu}{T}}\,x + D\sin\omega\sqrt{\frac{\mu}{T}}\,x$$

Boundary Conditions:

* at $x=0$, we have $y(0,t)=0 \Rightarrow X(0)=0 \Rightarrow 0 = C(1) + D(0) \Rightarrow C=0$

$$\therefore X(x) = D\sin\omega\sqrt{\frac{\mu}{T}}\,x$$

* at $x=L$, we have $y(L,t)=0 \Rightarrow X(L)=0 \Rightarrow 0 = D\sin\omega\sqrt{\frac{\mu}{T}}\,L$

for non trivial solution $D \neq 0$, hence $\sin\omega\sqrt{\frac{\mu}{T}}\,L = 0$

$$\therefore \omega_n\sqrt{\frac{\mu}{T}}\,L = n\pi \quad \Rightarrow \boxed{\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}} \quad \text{natural frequencies } (n=1,2,\dots)$$

$$\therefore y(x,t) = \sum_{n=1}^{\infty}(A_n\cos\omega_n t + B_n\sin\omega_n t)\sin\frac{n\pi x}{L}$$

$$\dot{y}(x,t) = \sum_{n=1}^{\infty}\omega_n(A_n\sin\omega_n t - B_n\cos\omega_n t)\sin\frac{n\pi x}{L}$$

① cont

Initial Conditions:

* at $t=0$, $\dot{y}(x,0) = 0 \Rightarrow 0 = \sum_{n=1}^{\omega} \omega_n \left(B_n(0) - A_n(1) \right) \sin \frac{n\pi x}{L} \Rightarrow A_n = \cdots$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} B_n \cos \omega_n t \sin \frac{n\pi x}{L}$$

* at $t=0$, $y(x,0) = y_0(x) = \frac{2d}{L} x \qquad$ for $0 \le x \le \frac{L}{2}$

$\qquad = \frac{2d}{L}[L-x] \qquad$ for $\frac{L}{2} \le x \le L$

and $\quad y_0(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$

multiplying the above equation by $\sin \frac{m\pi x}{L}$ & integrating between $0, L$

$$\therefore \int_0^L y_0(x) \sin \frac{m\pi x}{L} dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx$$

$$= B_n \frac{L}{2}$$

since $\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = 0$ for $m \ne n$

and $\int_0^L \sin^2 \frac{n\pi x}{L} dx = \frac{L}{2}$

$$\therefore B_m = \frac{2}{L} \int_0^L y_0(x) \sin \frac{m\pi x}{L} dx$$

$$= \frac{2}{L} \left\{ \int_0^{\frac{L}{2}} \frac{2d}{L} x \sin \frac{m\pi x}{L} dx + \int_{\frac{L}{2}}^{L} \frac{2d}{L} [L-x] \sin \frac{m\pi x}{L} dx \right\}$$

$$= \frac{4d}{L^2} \left\{ -\frac{L}{m\pi} \int_0^{\frac{L}{2}} x \, d\left(\cos \frac{m\pi x}{L} \right) + L \int_{\frac{L}{2}}^{L} \sin \frac{m\pi x}{L} dx + \frac{L}{m\pi} \int_{\frac{L}{2}}^{L} x \, d\left(\cos \frac{m\pi x}{L} \right) \right\}$$

$$= \frac{4d}{L^2} \left\{ -\frac{L}{m\pi} \left[x \cos \frac{m\pi x}{L} \Big|_0^{\frac{L}{2}} - \int_0^{\frac{L}{2}} \cos \frac{m\pi x}{L} dx \right] - \frac{L^2}{m\pi} \cos \frac{m\pi x}{L} \Big|_{\frac{L}{2}}^{L} + \frac{L}{m\pi} \left[x \cos \frac{m\pi x}{L} \Big|_{\frac{L}{2}}^{L} - \int_{\frac{L}{2}}^{L} \cos \frac{m\pi x}{L} dx \right] \right\}$$

$$= \frac{4d}{m\pi L} \left\{ \cancel{-\frac{L}{2} \cos \frac{m\pi}{2}} + \frac{L}{m\pi} \sin \frac{m\pi x}{L} \Big|_0^{\frac{L}{2}} \cancel{- L \cos m\pi} \cancel{+ L \cos \frac{m\pi}{2}} \cancel{+ L \cos m\pi} \cancel{- \frac{L}{2} \cos \frac{m\pi}{2}} - \frac{L}{m\pi} \sin \frac{m\pi x}{L} \Big|_{\frac{L}{2}}^{L} \right\}$$

$$= \frac{4d}{m^2\pi^2} \left\{ \sin \frac{m\pi}{2} - \cancel{\sin m\pi} + \sin \frac{m\pi}{2} \right\} = \frac{8d}{m^2\pi^2} \sin \frac{m\pi}{2}$$

$\therefore B_m = 0 \qquad$ if $m = \text{even} = 2, 4, 6, \ldots$

$\therefore B_m = (-1)^{\frac{m-1}{2}} \frac{8d}{\ldots} \qquad$ if $m = \text{odd} = 1, 3, 5, \ldots$

Rayleigh Ritz Method.

2

take the mode shape of an-uniform string for comparison function

$$u_1(x) = \sin\frac{\pi}{\ell}x$$

$$u_2(x) = \sin\frac{2\pi}{\ell}x$$

$$k_{11} = \int_D u_1 \, L[u_1] \, dD = T\left(\frac{\pi}{\ell}\right)^2 \int_0^\ell \sin\frac{\pi}{\ell}x \cdot \sin\frac{\pi}{\ell}x \, dx = T\frac{\pi^2}{2\ell}$$

$$k_{12} = \int_D u_1 \, L[u_2] \, dD = 4\left(\frac{\pi}{\ell}\right)^2 \int_0^\ell \sin\frac{\pi}{\ell}x \, \sin\frac{\pi 2}{\ell}x \, dx = 0$$

$$k_{22} = \int_D u_2 \, L[u_2] \, dD = 4\left(\frac{\pi}{\ell}\right)^2 \int_0^\ell \sin^2\frac{2\pi}{\ell}x \, dx = T\left(\frac{2\pi}{\ell}\right)^2 \frac{\ell}{2} = T\frac{2\pi^2}{\ell}$$

$$m_{11} = \int_D u_1 \, M[u_1] \, dD = \int_0^\ell \left[\mu_0 + \mu_1 x\left(1 - \frac{x}{\ell}\right)\right] \sin^2 x\frac{\pi}{\ell} \, dx$$

$$m_{11} = \frac{\ell}{2}\left[\mu_0 + \mu_1 \ell\left(\frac{1}{6} + \frac{1}{2\pi^2}\right)\right]$$

$$m_{12} = \int_D u_1 \, M[u_2] \, dD = \int_0^\ell \left[\mu_0 + \mu_1 x\left(1 - \frac{x}{\ell}\right)\right] \sin\frac{\pi x}{\ell} \sin\frac{2\pi x}{\ell} \, dx$$

$$m_{12} = \frac{\mu_1}{2}\left[\left(\frac{\ell}{\pi}\right)^2(-2) - \left(\frac{\ell}{3\pi}\right)^2(-2)\right] - \frac{\mu_1}{2\ell}\left[\left(\frac{\ell}{\pi}\right)^2(2) - \frac{\ell}{3\pi}\right.$$

$$m_{12} = 0$$

$$m_{22} = \int_D u_2 \, M[u_2] \, dD = \int_0^\ell \left[\mu_0 + \mu_1 x\left(1 - \frac{x}{\ell}\right)\right] \sin^2\frac{2\pi x}{\ell} \, dx.$$

$$m_{22} = \frac{\ell}{2}\left\{\mu_0 + \mu_1 \ell\left(\frac{1}{6} + \frac{1}{8\pi^2}\right)\right\}.$$

$$\underset{\sim}{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix} \qquad \underset{\sim}{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}$$

$$\left\{\underset{\sim}{K} - \lambda^{(n)}\underset{\sim}{M}\right\}\{\underset{\sim}{a}\} = \underset{\sim}{0}$$

2) cont'd

$$\left(\underset{\sim}{K} - \lambda^{(n)} \underset{\sim}{M}\right)\left(\underset{\sim}{a}\right) = \underset{\sim}{0}$$

$$\det\left(\underset{\sim}{K} - \lambda^{(n)} \underset{\sim}{M}\right) = 0$$

$$\begin{vmatrix} T\frac{\pi^2}{2l} - \lambda \frac{l}{2}\left[\mu_0 + \mu_1 l\left(\frac{1}{6} + \frac{1}{2\pi^2}\right)\right] & 0 \\ 0 & T\frac{2\pi^2}{l} - \lambda \frac{l}{2}\left[\mu_0 + \mu_1 l\left[\frac{1}{6} + \frac{1}{8\pi^2}\right]\right] \end{vmatrix} = 0$$

$$\lambda_1 = \frac{T\frac{\pi^2}{l\cancel{2}}}{\frac{l}{\cancel{2}}\left[\mu_0 + \mu_1 l\left(\frac{1}{6} + \frac{1}{2\pi^2}\right)\right]}$$

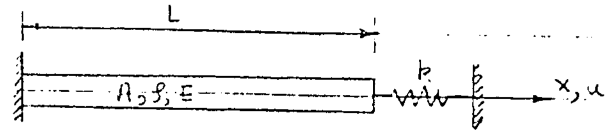
$$\lambda_2 = \frac{T\frac{2\pi^2}{l}}{\frac{l}{2}\left[\mu_0 + \mu_1 l\left(\frac{1}{6} + \frac{1}{8\pi^2}\right)\right]}$$

$$\boxed{\omega_1 = \frac{\pi}{l}\sqrt{\frac{T}{\mu_0 + \mu_1 l\left(\frac{1}{6} + \frac{1}{2\pi^2}\right)}} \qquad \text{Rayleigh Ritz}}$$

$$\omega_2 = \frac{2\pi}{l}\sqrt{\frac{T}{\mu_0 + \mu_1 l\left(\frac{1}{6} + \frac{1}{8\pi^2}\right)}}$$

(3)

(a) The longitudenal oscillation of bars is governed by the differential equation



$$\frac{\partial}{\partial x}\left[AE\frac{\partial u}{\partial x}\right] = \rho A \frac{\partial^2 u}{\partial t^2}$$

since the bar is uniform, i.e E, A & ρ = constant, then the above equation reduces to:-

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$$

Assuming a solution in the form:- $u(x,t) = X(x)\,.\,T(t)$

$\therefore\ T''(t)\,.\,X(x) = \frac{E}{\rho}\,X''(x)\,T(t)$

or $\frac{E}{\rho}\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}$ = constant $= -\omega^2$

$\therefore\ T''(t) + \omega^2 T(t) = 0 \quad \Rightarrow T(t) = A\sin\omega t + B\cos\omega t$

& $X''(x) + \frac{\rho\omega^2}{E}X(x) = 0 \quad \Rightarrow X(x) = C\sin(\omega\sqrt{\frac{\rho}{E}}\,x) + D\cos(\omega\sqrt{\frac{\rho}{E}}\,x)$

Boundary Conditions:

* at $x=0$, we have $u(0,t)=0 \quad \Rightarrow X(0) = 0$

  $\therefore\ 0 = C(0) + D(1) \Rightarrow D = 0$

  $\therefore\ X(x) = C\sin(\omega\sqrt{\frac{\rho}{E}}\,x)$

* at $x=\ell$, we have $F = -k\,u(\ell,t) = AE\left.\frac{\partial u}{\partial x}\right|_{x=\ell}$

  $\Rightarrow\ -k\,X(\ell) = AE\left.\frac{dX}{dx}\right|_{x=\ell}$

  $\therefore\ -k\,\cancel{C}\sin(\omega\sqrt{\frac{\rho}{E}}\,\ell) = AE\,\cancel{C}\,\omega\sqrt{\frac{\rho}{E}}\cos(\omega\sqrt{\frac{\rho}{E}}\,\ell)$

$\therefore$ the frequency equation is

$\boxed{\tan(\omega\sqrt{\rho}\,\ell) = -\ EA\omega\sqrt{\rho}}$

3 cont

[b] * for the system with the spring stiffness equal the bar longitudinal stiffness:

$$k = \frac{EA}{\ell}$$

substituting into (*) we get:-

$$\tan\left(\omega\sqrt{\tfrac{\rho}{E}}\,\ell\right) = -\frac{EA\,\omega}{\frac{EA}{\ell}}\sqrt{\frac{\rho}{E}} = -\omega\sqrt{\frac{\rho}{E}}\,\ell$$

or $\tan\theta = -\theta$ where $\theta = \omega\sqrt{\frac{\rho}{E}}\,\ell$

the solution of the above equation is $\theta = 2.029$ [1st solution]

$$\therefore \boxed{\omega_1 = \frac{2.029}{\ell}\sqrt{\frac{E}{\rho}}}$$ ✓

* for the bar alone:

$$k = 0$$

substituting into (*) we get:-

$$\tan\left(\omega\sqrt{\tfrac{\rho}{E}}\,\ell\right) = \infty \implies \omega\sqrt{\frac{\rho}{E}}\,\ell = \frac{\pi}{2}$$ [1st solution]

$$\therefore \boxed{\omega_1' = \frac{\pi}{2\ell}\sqrt{\frac{E}{\rho}}}$$

$\therefore$ the required frequency ratio

$\omega_1 = 2.029\sqrt{E} / \pi\sqrt{E} = \boxed{1.29}$ ✓