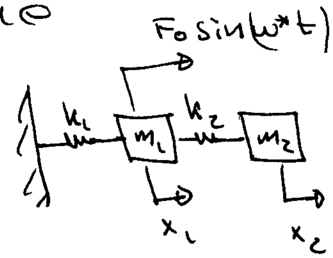
HW 4 ME588 Solutions

(1) The amplitude of response at the first mass m_1 is given by:



$\omega_*^2 = \frac{k_1}{m_1}$

$$X_1 = \frac{(k_2 - m_2\omega_*^2)F_0}{\Delta(\omega_*)}$$

where $\Delta(\omega_*) = \begin{vmatrix} k_1+k_2-m_1\omega_*^2 & -k_2 \\ -k_2 & k_2-m_2\omega_*^2 \end{vmatrix}$

$$= (k_1+k_2-m_1\omega_*^2)(k_2-m_2\omega_*^2) - k_2^2$$

$$= m_1m_2\omega_*^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega_*^2 + k_1k_2$$

$$= m_1m_2\left\{\omega_*^4 - \left[\frac{k_1}{m_1} + \frac{k_2}{m_1} + \frac{k_2}{m_2}\right]\omega_*^2 + \frac{k_1}{m_1}\cdot\frac{k_2}{m_2}\right\}$$

To find the natural frequencies we must solve the equation $\Delta(\omega) = 0 \Rightarrow$

$$\omega^4 - \left[\frac{k_1}{m_1} + \frac{k_2}{m_1} + \frac{k_2}{m_2}\right]\omega^2 + \frac{k_1}{m_1}\cdot\frac{k_2}{m_2} = 0 \qquad (1)$$

For minimum $X_1 \Rightarrow \omega_*^2 = \frac{k_2}{m_2}$, also given that $\omega_*^2 = \frac{k_1}{m_1}$

$\Rightarrow \frac{k_2}{m_1} = \frac{m_2}{m_1}\frac{k_2}{m_2} = \frac{m_2}{m_1}\cdot\omega_*^2$ this into (1) gives:

$$\omega^4 - \left[\omega_*^2 + \frac{m_2}{m_1}\omega_*^2 + \omega_*^2\right]\omega^2 + \omega_*^4 = 0$$

$$\omega^4 - \left[2 + \frac{m_2}{m_1}\right]\omega_*^2 + \omega_*^4 = 0$$

$$\Rightarrow \omega_{1,2}^2 = \frac{\omega_*^2}{2}\left\{\left[2+\frac{m_2}{m_1}\right] \mp \sqrt{\left(2+\frac{m_2}{m_1}\right)^2 - 4}\right\}$$

$$\omega_{1,2}^2 = \left\{\left[1 + \frac{1}{2}\frac{m_2}{m_1}\right] \mp \sqrt{\frac{1}{4}\left(\frac{m_2}{m_1}\right)^2 + \frac{m_2}{m_1}}\right\}\omega_*^2$$

$$\omega_{1,2}^2 = \left\{1 + \frac{1}{2}\frac{m_2}{m_1} \mp \frac{1}{2}\frac{m_2}{m_1}\sqrt{1 + 4\frac{m_1}{m_2}}\right\}\omega_*^2$$

① cont.

$\underline{\underline{\omega_1}}$:
$$\omega_1^2 = \left\{ 1 + \frac{1}{2}\frac{m_2}{m_1} - \frac{1}{2}\frac{m_2}{m_1}\sqrt{1 + 4\frac{m_1}{m_2}} \right\} \omega_*^2$$

since $\frac{m_1}{m_2} > 0 \Rightarrow \sqrt{1 + 4\frac{m_1}{m_2}} = 1 + \delta$ where $\delta > 0$

$$\Rightarrow \frac{\omega_1^2}{\omega_*^2} = 1 + \frac{1}{2}\frac{m_2}{m_1} - \frac{1}{2}\frac{m_2}{m_1}(1+\delta)$$
$$= 1 - \underbrace{\frac{1}{2}\frac{m_2}{m_1}\cdot\delta}_{\text{positive.}}$$

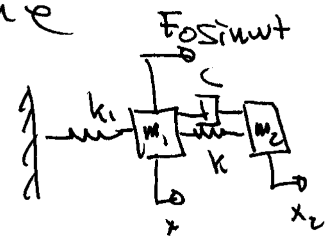
$$\Rightarrow \frac{\omega_1^2}{\omega_*^2} < 1 \Rightarrow \boxed{\omega_1 < \omega_*}$$

ω_2 :
$$\omega_2^2 = \left\{ 1 + \frac{1}{2}\frac{m_2}{m_1} + \frac{1}{2}\frac{m_2}{m_1}(1+\delta) \right\} \omega_*^2$$

$$\Rightarrow \frac{\omega_2^2}{\omega_*^2} = 1 + \underbrace{\frac{m_2}{m_1} + \frac{1}{2}\frac{m_2}{m_1}\delta}_{\text{positive}}$$

$$\Rightarrow \frac{\omega_2^2}{\omega_*^2} > 1 \Rightarrow \boxed{\omega_2 > \omega_*}$$

(2) The amplitude of response of the mass m_1 is given by:



$$\bar{X}_1 = \frac{F_0(k - m_2\omega^2 + i\omega c)}{(k_1 + k - m_1\omega^2 + i\omega)(k - m_2\omega^2 + i\omega c) - (k + i\omega c)^2}$$

$$= \frac{F_0[(k - m_2\omega^2) + i\omega c]}{[k_1 k - (m_1 k + m_2 k_1 + m_2 k)\omega^2 + m_1 m_2 \omega^4] + [k_1 - (m_1 + m_2)\omega^2] i\omega c}$$

For $c = 0$

$$\bar{X}_1 = \frac{F_0(k - m_2\omega^2)}{[k_1 k - (m_1 k + m_2 k_1 + m_2 k)\omega^2 + m_1 m_2 \omega^4]} \qquad (1)$$

For $c = \infty$

$$\bar{X}_1' = \frac{F_0}{k_1 - (m_1 + m_2)\omega^2} \qquad (2)$$

To find ω_a, ω_b we have

$$\bar{X}_1' = -\bar{X}_1$$

$$\Rightarrow \frac{F_0}{k_1 - (m_1 + m_2)\omega^2} = \frac{-F_0(k - m_2\omega^2)}{k_1 k - (m_1 k + m_2 k_1 + m_2 k)\omega^2 + m_1 m_2 \omega^4}$$

$$\Rightarrow (m_2^2 + 2m_1 m_2)\omega^4 - 2(m_1 k + m_2 k_1 + m_2 k)\omega^2 + 2kk_1 = 0$$

$$\Rightarrow \omega_{a,b}^2 = \left\{(m_1 k + m_2 k_1 + m_2 k) \mp \sqrt{(m_1 k + m_2 k_1 + m_2 k)^2 - 2kk_1(m_2^2 + 2m_1 m_2)}\right\} \Big/ (m_2^2 + 2m_1 m_2)$$

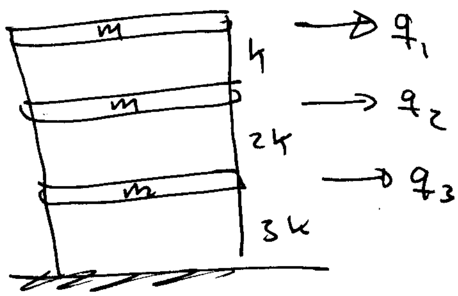
Now for the amplitude of response associated with each of the two points of ω_a, ω_b to be the same:

$$\bar{X}_1\Big|_{\omega=\omega_a} = -\bar{X}_1'\Big|_{\omega=\omega_b} \Rightarrow \frac{F_0/(m_1+m_2)}{\frac{k_1}{(m_1+m_2)} - \omega_a^2} = -\frac{F_0/(m_1+m_2)}{\frac{k_1}{(m_1+m_2)} - \omega_b^2}$$

$$\Rightarrow \frac{2k_1}{(m_1+m_2)} = \omega_a^2 + \omega_b^2 = \frac{2(m_1 k + m_2 k_1 + m_2 k)}{m_2^2 + 2m_1 m_2}$$

$$\Rightarrow m_2^2 k_1 + 2m_1 m_2 k = (m_1 + m_2)^2 + m_1 m_2 k_1 + m_2^2 k_1 \Rightarrow \boxed{\frac{k_1}{k_2} = \frac{m_1 m_2}{(m_1 + m_2)^2}}$$

(3)



a) Eigen value problem: $(H - I\omega^2)\underset{\sim}{X} = 0$

where $H = M^{-1}K$, $M = mI$, $K = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

$H = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} \frac{k}{m}$ let $\lambda = \omega^2 / (k/m)$

so non trivial sol if:

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 3-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 19\lambda^2 + 18\lambda - 6 = 0$$

$\Rightarrow \lambda_1 = 0.4156$, $\lambda_2 = 2.297$, $\lambda_3 = 6.29$

$\Rightarrow \omega_1 = 0.645\sqrt{\frac{k}{m}}$, ω_2 $1.52\sqrt{k/m}$, $\omega_3 = 2.51\sqrt{\frac{k}{m}}$

b) mode shapes

Eigenvectors: $(H - \lambda_i I)\underset{\sim}{X}^i = \underset{\sim}{0}$

$i=1$ $H - \lambda_1 I = \frac{k}{m}\begin{pmatrix} 0.5844 & -1 & 0 \\ -1 & 2.5844 & -2 \\ 0 & -2 & 4.5844 \end{pmatrix}\begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \end{pmatrix} = \underset{\sim}{0}$

$\Rightarrow$ Let $X_1^1 = 1$ $\Rightarrow$ $X_2^1 = 0.5844$, $X_3^1 = 0.2552$

$$\Rightarrow \underset{\sim}{X}^1 = \begin{pmatrix} 1 \\ 0.5844 \\ 0.2552 \end{pmatrix} \Longleftarrow$$

(3) cont

$\underline{\underline{i=2}}$

$$\frac{k}{m}\begin{pmatrix} -1.297 & -1 & 0 \\ -1 & 0.703 & -2 \\ 0 & -2 & 2.703 \end{pmatrix}\begin{pmatrix} \bar{X}_1^2 \\ \bar{X}_2^2 \\ \bar{X}_3^2 \end{pmatrix} = \underset{\sim}{0}$$

$\Rightarrow$ Let $\bar{X}_1^2 = 1 \Rightarrow \bar{X}_2^2 = -1.297$ (from 1st row)

$\bar{X}_3^2 = -0.956$ (from 2nd or 3rd row)

$$\underset{\sim}{\bar{X}}^2 = \begin{pmatrix} 1 \\ -1.297 \\ -0.956 \end{pmatrix} \Longleftarrow$$

$\underline{\underline{i=3}}$

$$\frac{k}{m}\begin{pmatrix} -5.29 & -1 & 0 \\ -1 & -3.29 & -2 \\ 0 & -2 & -1.29 \end{pmatrix}\begin{pmatrix} \bar{X}_1^3 \\ \bar{X}_2^3 \\ \bar{X}_3^3 \end{pmatrix} = \underset{\sim}{0}$$

$\Rightarrow$ Let $\bar{X}_1^3 = 1.0 \Rightarrow \bar{X}_2^3 = -5.29$ (from 1st row)

$\Rightarrow \bar{X}_3^3 = 8.202$

$$\underset{\sim}{\bar{X}}^3 = \begin{pmatrix} 1 \\ -5.29 \\ 8.202 \end{pmatrix} \Longleftarrow$$

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7.6. A beam of mass per unit length $m(x)$ and bending stiffness EI(x), free at both ends lies on an elastic foundation of distributed stiffness $k(x)$, as shown in Fig. 7.35. Derive the boundary-value problem for the bending vibration of the beam.

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Hamilton's Principle

From Hamilton's principle, we can write

$$\int_{t_1}^{t_2}(\delta T - \delta V + \overline{\delta W_{nc}})\, dt \qquad \textit{(Eq.1.1)}$$

$$\delta w(x, t_1) = \delta w(x, t_2) = 0 \qquad \textit{(Eq.1.2)}$$

We need to derive expressions for δT, δV, and $\overline{\delta W_{nc}}$.

Kinetic Energy

The kinetic energy is given by

$$T = \frac{1}{2}\int_0^L m(x)\left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx \qquad \textit{(Eq.1.3)}$$

We can now calculate the variation of T. To make the calculation easier, we can denote $\frac{\partial w(x,t)}{\partial t} = \dot{w}(x, t)$

$$\delta T = \delta\left[\frac{1}{2}\int_0^L m(x)\,\dot{w}(x, t)^2\, dx\right]$$

$$= \frac{1}{2}\int_0^L \delta[m(x)\,\dot{w}(x, t)^2]\, dx$$

$$= \frac{1}{2}\int_0^L\left\{\frac{\partial}{\partial m(x)}[m(x)\,\dot{w}(x, t)^2]\,\delta m(x) + \frac{\partial}{\partial \dot{w}(x,t)}[m(x)\,\dot{w}(x, t)^2]\,\delta \dot{w}(x, t)\right\} dx$$

Since we only allow $w(x, t)$ to vary, then $\delta m(x) = 0$

$$= \frac{1}{2}\int_0^L \frac{\partial}{\partial \dot{w}(x,t)}[m(x)\,\dot{w}(x, t)^2]\,\delta \dot{w}(x, t)\, dx$$

$$= \int_0^L m(x)\,\dot{w}(x, t)\,\delta\left[\frac{\partial}{\partial t}[w(x, t)]\right] dx$$

Now, if we assume that the $\delta[\,...]$ and $\frac{\partial}{\partial t}[\,...]$ operators are interchangable (see Appendix A), we can write

$$\delta T = \int_0^L m(x)\,\frac{\partial w(x,t)}{\partial t}\,\frac{\partial}{\partial t}[\delta w(x, t)]\, dx \qquad \textit{(Eq.1.4)}$$

Potential Energy

The potential energy is given by a combination of the bending moment which wants to restore the beam to its original position and the spring forces.

$$V = V_{\text{bend}} + V_{\text{spring}} \tag{Eq.1.5}$$

where $V_{\text{bend}} = \frac{1}{2}\int_0^L \mathrm{EI}(x)\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dx$ $\quad V_{\text{spring}} = \frac{1}{2}\int_0^L k(x)\, w(x,t)^2\, dx$

We can now calculate the variation in V. Let's start with the bending moment potential variation. To make the calculation easier, we can denote $\frac{\partial w(x,t)}{\partial x} = w'(x,t)$

$$\delta V_{\text{bend}} = \delta\left[\frac{1}{2}\int_0^L \mathrm{EI}(x)\, w''(x,t)^2\, dx\right]$$

$$= \frac{1}{2}\int_0^L \delta[\mathrm{EI}(x)\, w''(x,t)^2]\, dx$$

$$= \frac{1}{2}\int_0^L\left\{\frac{\partial}{\partial \mathrm{EI}(x)}[\mathrm{EI}(x)\, w''(x,t)^2]\,\delta\,\mathrm{EI}(x) + \frac{\partial}{\partial w''(x,t)}[\mathrm{EI}(x)\, w''(x,t)^2]\,\delta w''(x,t)\right\} dx$$

Once again, we only allow $w(x,t)$ to vary, so $\delta\,\mathrm{EI}(x) = 0$

$$= \frac{1}{2}\int_0^L \frac{\partial}{\partial w''(x,t)}[\mathrm{EI}(x)\, w''(x,t)^2]\,\delta w''(x,t)\, dx$$

$$= \int_0^L \mathrm{EI}(x)\, w''(x,t)\,\delta w''(x,t)\, dx$$

$$= \int_0^L \mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\,\delta\left[\frac{\partial^2}{\partial x^2}[w(x,t)]\right] dx$$

Once again, we switch the $\frac{\partial}{\partial x}[\ldots]$ and $\delta[\ldots]$ operator to obtain

$$\delta V_{\text{bend}} = \int_0^L \mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\,\frac{\partial^2}{\partial x^2}[\delta w(x,t)]\, dx \tag{Eq.1.6}$$

We can simplify this even further by using integration by parts. Namely, we have

$$\int_0^L u(x)\,\frac{\partial v(x)}{\partial x}\, dx = [u(x)\, v(x)]\Big|_0^L - \int_0^L v(x)\,\frac{du(x)}{dx}\, dx$$

so we choose

$$u(x) = \mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2} \quad\Rightarrow\quad \frac{\partial u(x)}{\partial x} = \frac{\partial}{\partial x}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right]$$

$$\frac{\partial v(x)}{\partial x} = \frac{\partial^2}{\partial x^2}[\delta w(x,t)] \Rightarrow \quad v(x) = \frac{\partial}{\partial x}[\delta w(x,t)]$$

Eq.1.6 becomes

$$\delta V_{\text{bend}} = \left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\,\frac{\partial}{\partial x}\,\delta w(x,t)\right]\Bigg|_0^L - \int_0^L \frac{\partial}{\partial x}[\delta w(x,t)]\,\frac{\partial}{\partial x}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] dx$$

Let us examine the integral more closely. As can seen, we need to integrate by parts once more. We can choose

$$u(x) = \frac{\partial}{\partial x}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] \quad \Rightarrow \quad \frac{\partial u(x)}{\partial x} = \frac{\partial^2}{\partial x^2}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right]$$

$$\frac{\partial v(x)}{\partial x} = \frac{\partial}{\partial x}[\delta w(x,t)] \;\Rightarrow\; v(x) = \delta w(x,t)$$

So the expression for δV_{bend} is given by

$$\delta V_{\text{bend}} = \left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\,\frac{\partial}{\partial x}\,\delta w(x,t)\right]\Bigg|_0^L - \left[\frac{\partial}{\partial x}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right]\delta w(x,t)\right]\Bigg|_0^L + \int_0^L \frac{\partial^2}{\partial x^2}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right]\delta w(x,t)\,dx \qquad (Eq.1.7)$$

Now for the spring, we have

$$\delta V_{\text{spring}} = \delta\left[\frac{1}{2}\int_0^L k(x)\,w(x,t)^2\,dx\right]$$

$$= \frac{1}{2}\int_0^L \delta[k(x)\,w(x,t)^2]\,dx$$

$$= \frac{1}{2}\int_0^L \left\{\frac{\partial}{\partial k(x)}[k(x)\,w(x,t)^2]\,\delta k(x) + \frac{\partial}{\partial w(x,t)}[k(x)\,w(x,t)^2]\,\delta w(x,t)\right\} dx$$

Once again, we only allow $w(x,t)$ to vary, so $\delta k(x) = 0$

$$= \frac{1}{2}\int_0^L \frac{\partial}{\partial w(x,t)}[k(x)\,w(x,t)^2]\,\delta w(x,t)\,dx$$

$$\delta V_{\text{spring}} = \int_0^L k(x)\,w(x,t)\,\delta w(x,t)\,dx \qquad (Eq.1.8)$$

So the total variation in potential energy is

$$\delta V = \delta V_{\text{bend}} + \delta V_{\text{spring}}$$

$$= \left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\,\frac{\partial}{\partial x}\,\delta w(x,t)\right]\Bigg|_0^L - \left[\frac{\partial}{\partial x}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right]\delta w(x,t)\right]\Bigg|_0^L$$
$$+ \int_0^L \frac{\partial^2}{\partial x^2}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right]\delta w(x,t)\,dx + \int_0^L k(x)\,w(x,t)\,\delta w(x,t)\,dx$$

$$\delta V = \left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\,\frac{\partial}{\partial x}\,\delta w(x,t)\right]\Bigg|_0^L - \left[\frac{\partial}{\partial x}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right]\delta w(x,t)\right]\Bigg|_0^L + \int_0^L \left(\frac{\partial^2}{\partial x^2}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] + k(x)\,w(x,t)\right)\delta w(x,t)\,dx$$

Nonconservative Work

The nonconservative work is simply given by

$$\overline{W_{nc}} = \int_0^L f(x,t)\, w(x,t)\, \boldsymbol{dx} \qquad \textit{(Eq.1.9)}$$

We can now calculate the variation in $\overline{W_{nc}}$

$$\overline{\delta W_{nc}} = \delta\left[\int_0^L f(x,t)\, w(x,t)\, \boldsymbol{dx}\right]$$

$$= \int_0^L \delta[f(x,t)\, w(x,t)]\, \boldsymbol{dx}$$

$$= \int_0^L \frac{\partial}{\partial f(x,t)}[f(x,t)\, w(x,t)]\, \delta f(x,t) + \frac{\partial}{\partial w(x,t)}[f(x,t)\, w(x,t)]\, \delta w(x,t)\, \boldsymbol{dx}$$

Once again, we only allow $w(x,t)$ to vary, so $\delta f(x,t) = 0$

$$= \int_0^L \frac{\partial}{\partial w(x,t)}[f(x,t)\, w(x,t)]\, \delta w(x,t)\, \boldsymbol{dx}$$

$$\overline{\delta W_{nc}} = \int_0^L f(x,t)\, \delta w(x,t)\, \boldsymbol{dx} \qquad \textit{(Eq.1.10)}$$

Hamilton's Principle

We can now insert the expressions for δT, δV, and $\overline{\delta W_{nc}}$ into the Hamilton's principle equation (Eq.1.1) to obtain

$$\int_{t_1}^{t_2} (\delta T - \delta V + \overline{\delta W_{nc}})\, dt = 0$$

$$\int_{t_1}^{t_2} \delta T\, dt - \int_{t_1}^{t_2} \delta V\, dt + \int_{t_1}^{t_2} \overline{\delta W_{nc}}\, dt = 0$$

(Eq.1.11)

1st term

Let's evaluate the first integral

$$\int_{t_1}^{t_2} \delta T\, dt = \int_{t_1}^{t_2} \int_0^L m(x) \frac{\partial w(x,t)}{\partial t} \frac{\partial}{\partial t}[\delta w(x, t)]\, dx\, dt$$

We can switch the order of integration to obtain

$$= \int_0^L \int_{t_1}^{t_2} m(x) \frac{\partial w(x,t)}{\partial t} \frac{\partial}{\partial t}[\delta w(x, t)]\, dt\, dx$$

We can perform the inner integral w.r.t. t using integration by parts by choosing

$$u(t) = m(x) \frac{\partial w(x,t)}{\partial t} \quad \Rightarrow \quad \frac{\partial u(t)}{\partial t} = m(x) \frac{\partial^2 w(x,t)}{\partial t^2}$$

$$\frac{\partial v(t)}{\partial t} = \frac{\partial}{\partial t}[\delta w(x, t)] \quad \Rightarrow \quad v(t) = \delta w(x, t)$$

Therefore, we have

0

$$= \int_0^L \left\{ \cancel{\left[m(x) \frac{\partial w(x,t)}{\partial t} \delta w(x, t)\right] \Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2} m(x) \frac{\partial^2 w(x,t)}{\partial t^2} \delta w(x, t)\, dt \right\} dx$$

However, by Eq.1.2, we know that that variation at the end times must be zero, so $\delta(w, t_1) = \delta(w, t_2) = 0$ and we have (also switching the order of integration)

$$\int_{t_1}^{t_2} \delta T\, dt = -\int_{t_1}^{t_2} \int_0^L m(x) \frac{\partial^2 w(x,t)}{\partial t^2} \delta w(x, t)\, dx\, dt$$

(Eq.1.12)

2nd term

We can now evaluate the second integral

$$\int_{t_1}^{t_2} \delta V \, dt =$$

$$\int_{t_1}^{t_2} \left\{ \left[\mathrm{EI}(x) \frac{\partial^2 w(x,t)}{\partial x^2} \frac{\partial}{\partial x} \delta w(x,t) \right] \Bigg|_0^L - \left[\frac{\partial}{\partial x} \left[\mathrm{EI}(x) \frac{\partial^2 w(x,t)}{\partial x^2} \right] \delta w(x,t) \right] \Bigg|_0^L + \int_0^L \left(\frac{\partial^2}{\partial x^2} \left[\mathrm{EI}(x) \frac{\partial^2 w(x,t)}{\partial x^2} \right] + k(x) w(x,t) \right) \delta w(x,t) \, dx \right\} dt$$

(Eq.1.13)

This is already as simplified as it can be.

3rd term

We can now evaluate the third integral

$$\int_{t_1}^{t_2} \overline{\delta \mathrm{W}_{\mathrm{nc}}} \, dt = \int_{t_1}^{t_2} \int_0^L f(x,t) \, \delta w(x,t) \, dx \, dt$$

(Eq.1.14)

This is already as simplified as it can be.

Form Equation of Motion

We can now substitute Eq.1.12, Eq.1.13, and Eq.1.14 into Eq.1.11 to obtain

$$\int_{t_1}^{t_2} \delta T\, dt - \int_{t_1}^{t_2} \delta V\, dt + \int_{t_1}^{t_2} \overline{\delta W_{nc}}\, dt = 0$$

$$-\int_{t_1}^{t_2} \delta T\, dt + \int_{t_1}^{t_2} \delta V\, dt - \int_{t_1}^{t_2} \overline{\delta W_{nc}}\, dt = 0$$

$$\int_{t_1}^{t_2}\int_0^L m(x)\,\frac{\partial^2 w(x,t)}{\partial t^2}\,\delta w(x,t)\,dx\,dt +$$

$$\int_{t_1}^{t_2}\left\{\mathrm{BCs} + \int_0^L\left(\frac{\partial^2}{\partial x^2}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] + k(x)\,w(x,t)\right)\delta w(x,t)\,dx\right\}dt - \int_{t_1}^{t_2}\int_0^L f(x,t)\,\delta w(x,t)\,dx\,dt = 0$$

where $\mathrm{BCs} = \left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\,\frac{\partial}{\partial x}\,\delta w(x,t)\right]\Big|_0^L - \left[\frac{\partial}{\partial x}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right]\delta w(x,t)\right]\Big|_0^L$

$$\int_{t_1}^{t_2}\left\{\mathrm{BCs} + \int_0^L\left[\frac{\partial^2}{\partial x^2}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] + k(x)\,w(x,t) + m(x)\,\frac{\partial^2 w(x,t)}{\partial t^2} - f(x,t)\right]\delta w(x,t)\,dx\right\}dt = 0 \qquad \textit{(Eq.1.15)}$$

We can now invoke the arbitrariness of the variation in $\delta w(x, t)$. Therefore, the only way for the inner integral to be zero is if the coefficent of $\delta w(x, t)$ is identically equivalent to zero for all x and t. Therefore

$$\boxed{\frac{\partial^2}{\partial x^2}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] + k(x)\,w(x,t) + m(x)\,\frac{\partial^2 w(x,t)}{\partial t^2} - f(x,t) = 0} \qquad \textit{(Eq.1.16)}$$

In addition to this, we also require that the term outside the integral w.r.t. x be zero at as well. Therefore, this term is given by

$$\boxed{\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\,\frac{\partial}{\partial x}\,\delta w(x,t)\right]\Big|_0^L - \left[\frac{\partial}{\partial x}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right]\delta w(x,t)\right]\Big|_0^L = 0}$$

Since we know the EI(0) and EI(L) cannot be zero, we can form the boundary conditions. For the end at $x = 0$ we need either

$$\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2} = 0 \qquad \text{or} \qquad \delta w(x,t) = 0 \;\Rightarrow\; w(x,t) = 0 \qquad (\text{at } x = 0) \qquad \textit{(Eq.1.17)}$$

and also we require that either

$$\frac{\partial}{\partial x}\left[\mathrm{EI}(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] \qquad \text{or} \qquad \delta\,\frac{\partial w(0,t)}{\partial x} = 0 \;\Rightarrow\; \frac{\partial w(0,t)}{\partial x} = 0 \quad (\text{at } x = 0) \qquad \textit{(Eq.1.18)}$$

And similarly for the end at $x = L$.

Note that we could have ignored the spring force in the formulation of the potential energy and then later, we could have called this force a non-conservative force and inserted $f(x, t) = -k(x)\,w(x, t)$ in at Eq.1.16. We would then perform the integration if necessary to obtain any boundary conditions and we would obtain Eq.1.15 once again.