3.9

$y(x) = A\sin\left(\frac{2\pi x}{L}\right)$

$y(t) = A\sin\left(\frac{2\pi v t}{L}\right) = A\sin(\omega t)$ where $\omega = \frac{2\pi V}{L}$

system response and force transmitted?

Eq of motion: $m\ddot{z} + k(z-y) + c(\dot{z}-\dot{y}) = 0 \Rightarrow$

$$m\ddot{z} + c\dot{z} + kz = ky + c\dot{y}$$

or $$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \omega_n^2 y + 2\zeta\omega_n\dot{y}$$

Let $y(t) = Ae^{i\omega t}$ then $z = Z_1 e^{i\omega t} \Rightarrow$

$$(-\omega^2 + 2\zeta\omega_n\omega i + \omega_n^2) Z_1 = (2\zeta\omega_n\omega i + \omega_n^2) A$$

$$Z_1 = \frac{\omega_n^2 + 2\zeta\omega_n\omega i}{\omega_n^2 - \omega^2 + 2\zeta\omega_n\omega i} \cdot A$$

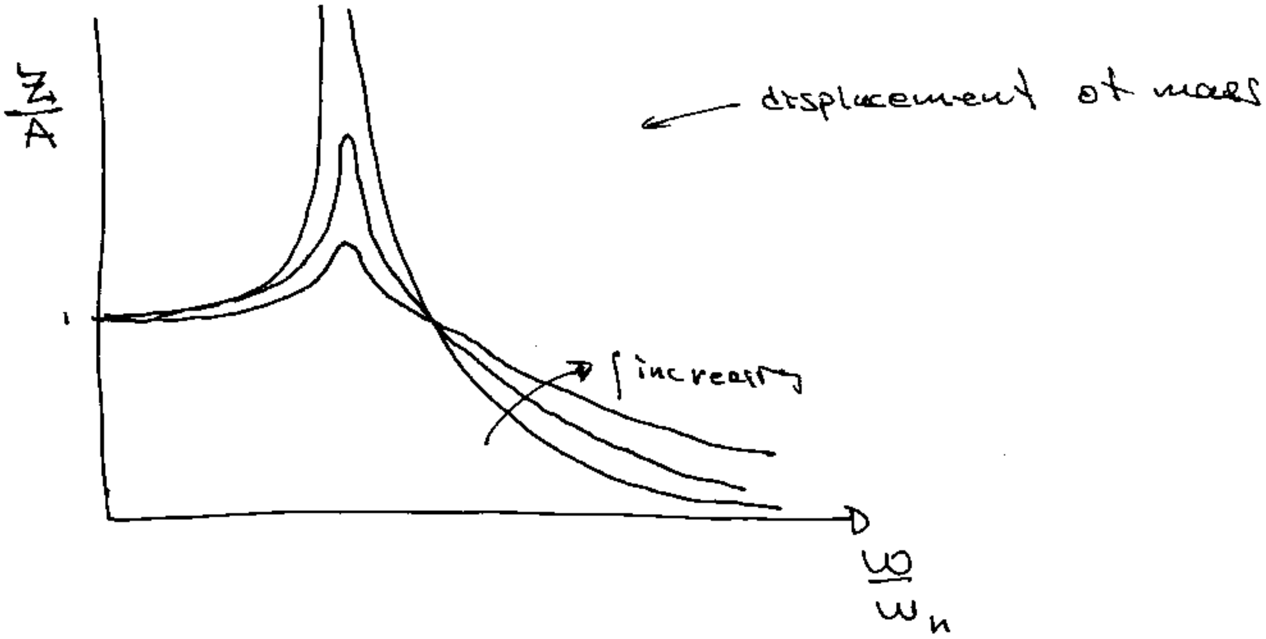
so $$z = \frac{\omega_n^2 + 2\zeta\omega_n\omega i}{\omega_n^2 - \omega^2 + 2\zeta\omega_n\omega i} \cdot A \cdot e^{i\omega t}$$

$$z = \frac{\left[\omega_n^4 + (2\zeta\omega_n\omega)^2\right]^{1/2} \cdot e^{i\theta_N}}{\left[(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2\right]^{1/2} \cdot e^{i\theta_D}} A \cdot e^{i\omega t}$$

where $\theta_N = \tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2}$, $\theta_D = \tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}$

$$z = \frac{\left[1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]^{1/2}}{\left[\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]^{1/2}} \cdot e^{i(\omega t + \theta_N - \theta_D)}$$

plot:



3.9 cont/ Now the force transmitted to the mass

Force transmitted $= F_T = F_{spring} + F_{damper}$

but this is also equal to $m\ddot{z}$ (since $m\ddot{z} + k(z-y) + c(\dot{z}-\dot{y}) = 0$)

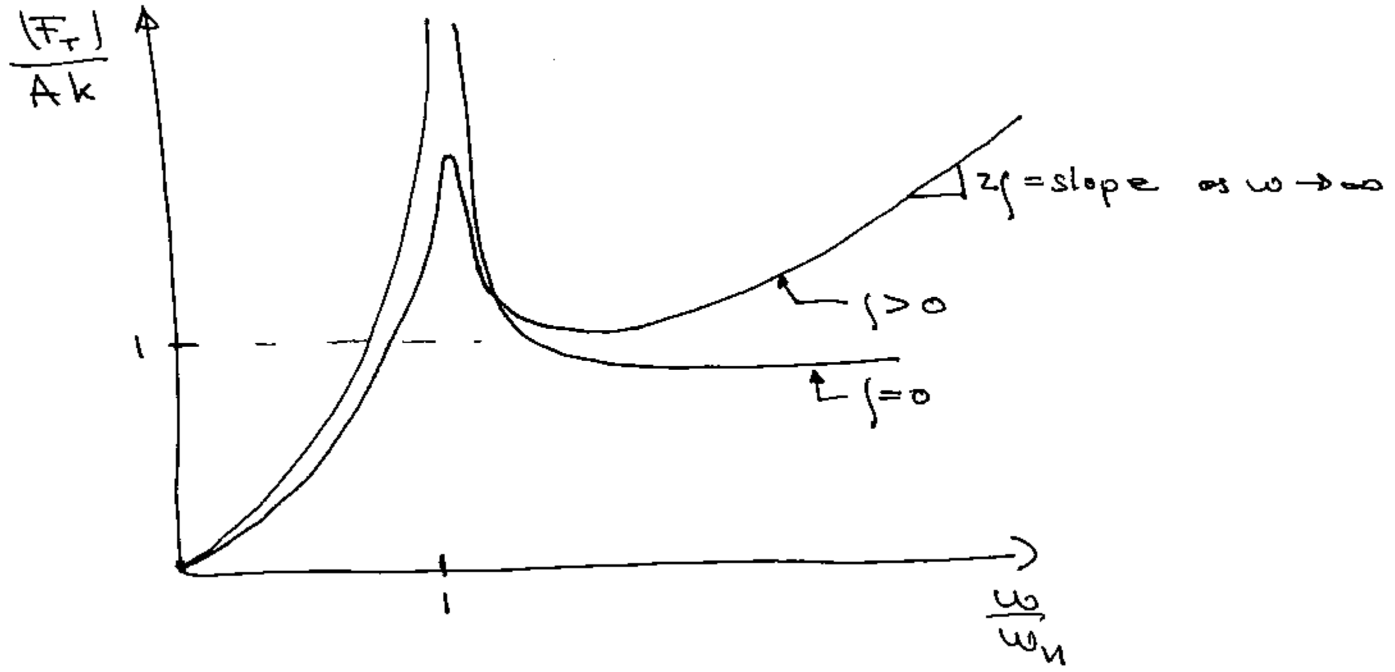
so $F_T = m\ddot{z} = -Am\omega^2 \dfrac{\left[1+\left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]^{1/2}}{\left[\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2\right]^{1/2}} e^{i(\omega t+\theta_N-\theta_D)}$

as $\omega \to 0 \Rightarrow |F_T| = 0$

as $\omega \to \infty \Rightarrow |F_T| = Am\omega^2 2\zeta\left(\frac{\omega}{\omega_n}\right) \Big/ \left(\frac{\omega}{\omega_n}\right)^2$, if $\zeta > 0$

$= 2Am\zeta\omega_n \cdot \omega = 2Ak\zeta \cdot \omega$, if $\underline{\underline{\zeta > 0}}$

plot: if $\underline{\underline{\zeta = 0}}$ then $F_T = A \cdot k$ as $\omega \to \infty$



so the force transmitted increases to ∞
as ω (or V) $\to \infty$
(except for when $\zeta = 0$)

3.10

Determine viscous damping factor so disp is $3e\frac{M}{m}$ max.

Spring constant $= \frac{L^2}{48EI} = k$

Eq of motion:

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin\omega t$$

$$x = \frac{1}{\omega_n^2}\frac{me\omega^2/M}{\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]^{1/2}} \cdot \sin(\omega t - \theta_D) \quad \text{where } \theta_D = \tan^{-1}\frac{2\zeta\,\omega/\omega_n}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

If $\zeta \ll 1$ then $\omega_p \approx \omega_n$ so the peak displacement occurs at $\omega = \omega_n$ $\Rightarrow$

$$X_{peak} \cong \frac{me/M}{2\zeta} = 3e\frac{M}{m} \Rightarrow$$

$$\zeta = \frac{m^2}{6M^2} \Rightarrow \frac{c}{2\sqrt{k \cdot M}} = \frac{m^2}{6M^2} \Rightarrow \boxed{c = \frac{2\sqrt{kM}\cdot m^2}{6M^2}}$$

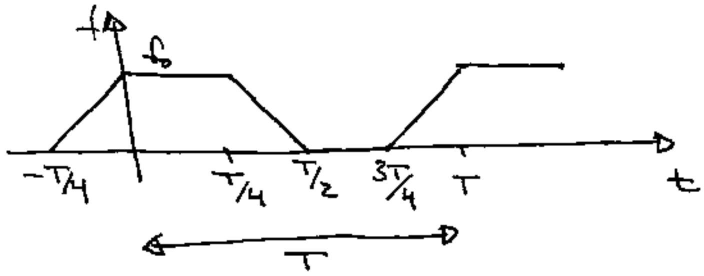
Note:

This problem makes more sense if the maximum displacement should be $3e\frac{m}{M}$ then:

$$X_{peak} \cong \frac{me/M}{2\zeta} = 3e\frac{m}{M} \Rightarrow$$

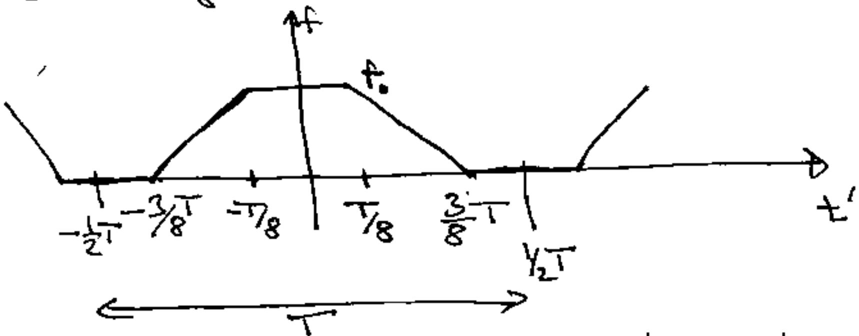
$$\boxed{\zeta = \frac{1}{6}}$$

3.12



to make this function even lets shift the time axis

$t' = t - T/8$



Lets drop the prime on t' and call it just "t".

$$f(t) = \begin{cases} 0 & -\frac{1}{2}T < t \le -3/8\,T \\ \frac{(t+\frac{3}{8}T)}{T} 4 \cdot f_0 & -\frac{3}{8}T < t \le -T/8 \\ f_0 & -T/8 < t \le T/8 \\ \frac{4 \cdot f_0}{T}(t - \frac{3}{8}T) & T/8 < t \le \frac{3}{8}T \\ 0 & \frac{3}{8}T < t \le \frac{1}{2}T \end{cases}$$

even function so $b_n = 0 \quad n = 1, 2 \ldots$

$$f(t) = \frac{1}{2}a_0 + a_1 \cos \omega t + a_2 \cos \omega t + \ldots$$

$$a_n = \frac{2}{T}\int_0^T f(t) \cdot \cos(n\omega t)\,dt = \frac{2}{T}\int_0^T f(t) \cdot \cos\left(n\frac{2\pi}{T} \cdot t\right)dt$$

$$a_n = \frac{2}{T}\left\{ \int_{-\frac{3}{8}T}^{-T/8} \frac{4f_0(t+\frac{3}{8}T)}{T} \cdot \cos\left(\frac{n 2\pi}{T} \cdot t\right)dt + \int_{-T/8}^{T/8} f_0 \cos\left(\frac{n 2\pi}{T} \cdot t\right)dt \right.$$
$$\left. + \int_{T/8}^{3/8\,T} \frac{4 f_0}{T}\left(t - \frac{3}{8}T\right)\cos\left(\frac{n 2\pi}{T} \cdot t\right)dt \right\}$$

integrate to get a_n $\quad n = 0, 1, 2 \ldots$ ☹

3.12 cont/

$$f(t) = \frac{1}{2}a_0 + a_1\cdot\cos\frac{2\pi}{T}\cdot t + a_2\cos\frac{4\pi}{T}t + a_3\cos\frac{6\pi}{T}\cdot t + \dots$$

Response of a first order system

$$\dot{x} + \beta\cdot x = f(t) = F_0\cdot e^{i\omega t}$$

$$x = \frac{F_0}{\beta + i\omega}\cdot e^{i\omega t} = \frac{F_0}{[\beta^2+\omega^2]^{1/2}}\, e^{i(\omega t - \phi)}$$

where $\phi = \tan^{-1}\left(\frac{\omega}{\beta}\right)$

(or for when the input is a cosine

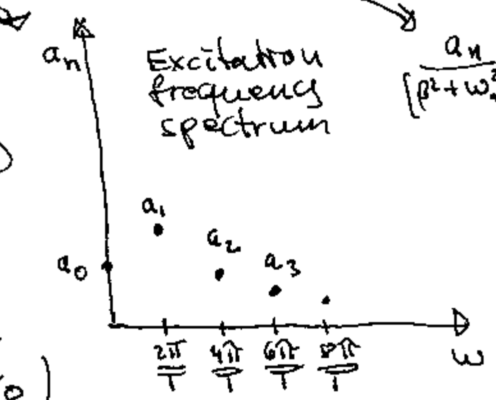
$$x = \frac{F_0}{(\beta^2+\omega^2)^{1/2}}\cdot\cos(\omega t - \phi) \quad)$$

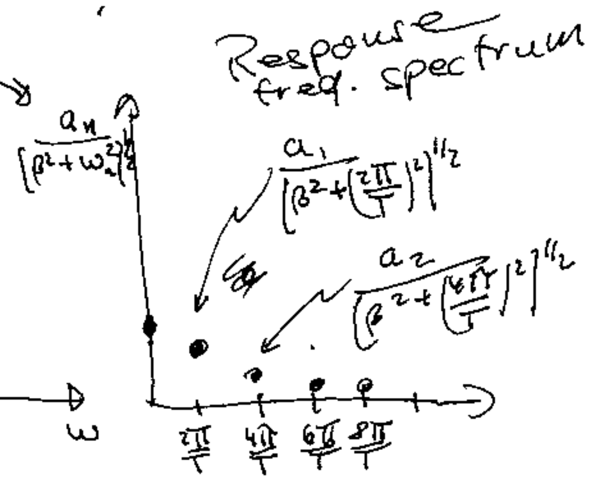
so for each term in the Fourier series:

ω_n	a_n	$\frac{a_n}{[\beta^2+\omega_n^2]^{1/2}}$	ϕ_n	x_n
0	a_0	$\frac{a_0}{[\beta^2+0]^{1/2}}$	0	$\frac{1}{2}\cdot a_0/\beta$
$\frac{2\pi}{T}$	a_1	$\frac{a_1}{[\beta^2+[\frac{2\pi}{T}]^2]^{1/2}}$	$\tan^{-1}\frac{2\pi/T}{\beta}$	$\frac{a_1}{[\beta^2+(\frac{2\pi}{T})^2]^{1/2}}\cdot\cos(\frac{2\pi}{T} - \phi_1)$
$\frac{4\pi}{T}$	a_2	$\frac{a_2}{[\beta^2+(\frac{4\pi}{T})^2]^{1/2}}$	$\tan^{-1}\frac{4\pi/T}{\beta}$	$\frac{a_2}{(\beta^2+(\frac{4\pi}{T})^2)^{1/2}}\cos(\frac{4\pi}{T} - \phi_2)$
$\frac{6\pi}{T}$	a_3	$\frac{a_3}{[\beta^2+(\frac{6\pi}{T})^2]^{1/2}}$	$\tan^{-1}\frac{6\pi/T}{\beta}$	$\frac{a_3}{(\beta^2+(\frac{6\pi}{T})^2)^{1/2}}\cdot\cos(\frac{6\pi}{T} - \phi_3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

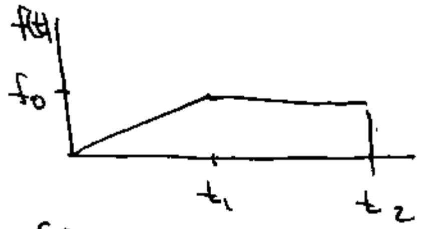
$$\boxed{x = \sum_{n=1}^{\infty} x_n}$$

(Now one can transfer back to the original time re. $t' = t - T/8$)





3.23



$$f(t)=\begin{cases}\dfrac{f_0}{t_1}\cdot t & 0<t\le t_1\\ f_0 & t_1<t<t_2\end{cases}$$

$$\begin{cases}x(0)=0\\ \dot{x}(0)=0\end{cases}$$

$\zeta=0.1 \quad \omega_n=4$

$m=1,\ t_1=1,\ t_2=2$

system response

$$x(t)=\left[\frac{\dot{x}(0)+\cancel{\zeta\omega_n x(0)}^{\,0}}{\omega_d}\cdot\sin\omega_d t+\cancel{x(0)\cos\omega_d t}^{\,0}\right]e^{-\zeta\omega_n t}+$$

$$+\frac{1}{m\omega_d}\int_0^t f(\tau)\,e^{-\zeta\omega_n(t-\tau)}\cdot\sin\omega_d(t-\tau)\,d\tau$$

① $$x(t)=\frac{1}{m\omega_d}\int_0^t\left(\frac{f_0}{t_1}\cdot\tau\right)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau \quad \text{for } \underline{0<t\le t_1}$$

② $$x(t)=\frac{1}{m\omega_d}\left[\int_0^{t_1}\left(\frac{f_0}{t_1}\cdot\tau\right)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau+\int_{t_1}^{t}f_0\,e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau\right] \quad \text{for } t_1<t<t_2$$

③ $$x(t)=\frac{1}{m\omega_d}\left[\int_0^{t_1}\left(\frac{f_0}{t_1}\cdot\tau\right)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau+\int_{t_1}^{t_2}f_0\,e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau\right] \quad \text{for } t>t_2$$

<u>Note</u>: integrate with respect to $\underline{\tau}$

(3.xx) $C_{eq} = \frac{\Delta W_d}{\pi \omega A^2}$ (from class notes)

Let $x \cong A\cos\theta$

$\dot{x} \cong -A\omega\sin\theta$ where $\theta = \omega t - \phi$

$(d\theta = \omega \cdot dt)$

$$\Delta W_d = \int_0^{2\pi/\omega} \underbrace{(a\dot{x}|\dot{x}|)}_{F_c} \cdot \dot{x}\, dt = \int_0^{\pi} \frac{a(A\omega)^3}{\omega} \sin^3\theta\, d\theta - \int_{\pi}^{2\pi} \frac{a(A\omega)^3}{\omega} \sin^3\theta\, d\theta$$

$$\Delta W_d = aA^3\omega^2 \left\{ -\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi} + \left[\cos\theta - \frac{\cos^3\theta}{3}\right]_{\pi}^{2\pi} \right\}$$

$$\Delta W_d = aA^3\omega^2 \frac{8}{3}$$

$$\Rightarrow C_{eq} = \frac{8aA\omega}{3\pi}$$

$$AF \cong \frac{1}{\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]^{1/2}} = \frac{1}{\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left[\frac{C_{eq}\cdot\omega}{k}\right]^2\right]^{1/2}}$$

$$= \frac{1}{\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left[\frac{8aA\omega^2}{3\pi k}\right]^2\right]^{1/2}}$$

At resonance $\omega \to \sim\omega_n$ (for small damping)

$$\Rightarrow AF = \frac{1}{\frac{8aA\omega^2}{3\pi k}}$$

but $AF = \frac{A}{F_0/k} = \frac{3\pi k}{8aA\omega_n^2}$

$$\Rightarrow \boxed{A = \frac{3\pi F_0}{8a\omega_n^2}}$$ ⇐ amplitude at resonance