Love Rayleigh Bar

Hamilton's Principle :

$$\delta\int_{t_1}^{t_2}(T-V)\,dt=0$$

$$\Rightarrow \delta\int_{t_1}^{t_2}\left\{\frac{\rho A}{2}\int_0^{\ell}\left[\dot u^2+\nu^2 r_g^2\left(\frac{\partial \dot u}{\partial z}\right)^2\right]dz-\frac{EA}{2}\int_0^{\ell}\left(\frac{\partial u}{\partial z}\right)^2dz\right\}dt=0$$

$$\Rightarrow \int_{t_1}^{t_2}\int_0^{\ell}\left\{\rho\left[\underbrace{\delta(\dot u^2)}_{1}+\underbrace{\nu^2 r_g^2\,\delta\left(\left(\frac{\partial\dot u}{\partial z}\right)^2\right)}_{2}\right]-\underbrace{E\,\delta\left(\left(\frac{\partial u}{\partial z}\right)^2\right)}_{3}\right\}dt\,dz=0$$

First term.

$$\int_{t_1}^{t_2}\delta(\dot u^2)\,dt=2\int_{t_1}^{t_2}\dot u\,\delta(\dot u)\,dt=2\int_{t_1}^{t_2}\dot u\,\frac{\partial}{\partial t}(\delta u)\,dt$$

$$=2\left\{\cancelto{0}{\left[\dot u\,\delta u\right]_{t_1}^{t_2}}-\int_{t_1}^{t_2}\ddot u\cdot\delta u\,dt\right\}$$

Third term

$$\int_0^{\ell}\delta\left(\frac{\partial u}{\partial z}\right)^2dz=2\int_0^{\ell}\frac{\partial u}{\partial z}\,\delta\left(\frac{\partial u}{\partial z}\right)dz=2\int_0^{\ell}\frac{\partial u}{\partial z}\,\frac{\partial}{\partial z}(\delta u)\,dz$$

$$=2\left\{\frac{\partial u}{\partial z}\cdot\delta u\Big|_0^{\ell}-\int_0^{\ell}\frac{\partial^2 u}{\partial z^2}\,\delta u\,dz\right\}$$

Second term

$$\int_0^{\ell}\int_{t_1}^{t_2} \delta\left(\frac{\partial \dot{u}}{\partial z}\right)^2 dt\, dz = 2\int_0^{\ell}\int_{t_1}^{t_2} \frac{\partial \dot{u}}{\partial z}\cdot\frac{\partial}{\partial z}(\delta \dot{u})\, dz\, dt$$

$$= 2\int_0^{\ell}\int_{t_1}^{t_2} \frac{\partial \dot{u}}{\partial z}\cdot\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}(\delta u)\right) dz\, dt$$

$$= 2\int_0^{\ell}\left\{\cancel{\left[\frac{\partial \dot{u}}{\partial z}\frac{\partial}{\partial z}(\delta u)\right]_{t_2}^{t_1}}^{\,0} - \int_{t_1}^{t_2} \frac{\partial \ddot{u}}{\partial z}\frac{\partial}{\partial z}(\delta u)\, dt\right\} dz$$

$$= -2\int_{t_1}^{t_2}\left\{\left[\frac{\partial \ddot{u}}{\partial z}\cdot\delta u\right]_0^{\ell} - \int_0^{\ell} \frac{\partial^2 \ddot{u}}{\partial z^2}\,\delta u\, dz\right\} dt$$

Putting all this into our equation $\Rightarrow$

$$\int_0^{\ell}\int_{t_1}^{t_2} \left\{\rho\left[-\ddot{u} + \nu^2 r_g^2 \frac{\partial^2 \ddot{u}}{\partial z^2}\right] + E\frac{\partial^2 u}{\partial z^2}\right\}\delta u\, dt\, dz$$

$$- \int_{t_1}^{t_2}\left[\left(E\frac{\partial u}{\partial z} + \frac{\partial \ddot{u}}{\partial z}\right)\delta u\right]_0^{\ell} dt = 0$$

since δu is arbitrary over the time interval $\Rightarrow$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\rho}{E}\left[\ddot{u} - \nu^2 r_g^2 \frac{\partial^2 \ddot{u}}{\partial z^2}\right]$$

$$\frac{\rho}{E} = \frac{1}{c^2} \quad \text{where} \quad c = \sqrt{\frac{E}{\rho}} \text{ - wave speed}$$