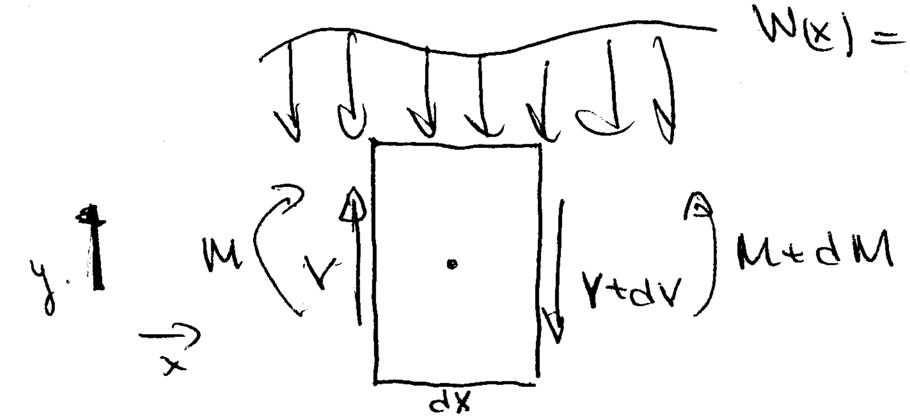
Lateral Vibration of a Beam

$W(x)$ = load / length



statics: $\Sigma F_y = 0 \quad \Rightarrow \quad + W\,dx + dV = 0$

$$\Rightarrow \quad W(x) = -\frac{dV}{dx}$$

$$V \cdot \frac{dx}{2} + (V + \cancel{dV}^{\,0})\frac{dx}{2} = dM$$

$$\Sigma M_{cm} = 0 \quad \Rightarrow \quad V \cdot dx = dM$$

$$\Rightarrow \quad V(x) = \frac{dM}{dx}$$

so we get $\quad W(x) = -\dfrac{d^2 M}{dx^2} \qquad (1)$

From strength of materials:

Plane x-section remain plane:

$$M = -EI\frac{d^2 y}{dx^2} \qquad (2)$$

(1) and (2) $\Rightarrow$

$$W(x) = -\frac{d^2 M}{dx^2} = -\frac{d^2}{dx^2}\left[EI\frac{d^2 y}{dx^2}\right]$$

Free Vibration: No external load.

$$W(x,t) = \rho A \frac{\partial^2 y}{\partial t^2}$$

Hence

$$\boxed{\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 y}{\partial x^2}\right] + \rho A\frac{\partial^2 y}{\partial t^2} = 0}$$

Let EI be constant

$$\boxed{EI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = 0}$$

To generate a soln, use sep. of variable.

Let. $y(x,t) = X(x)\, T(t)$. $\Rightarrow$

$$\frac{d^4 X}{dx^4} T(t) + \frac{\rho A}{EI} \frac{d^2 T}{dt^2} \cdot X(x) = 0$$

$$\underbrace{-\frac{T(t)}{\frac{d^2 T}{dt^2}}}_{f(t)} = \underbrace{\frac{\rho A}{EI} \frac{X(x)}{\frac{d^4 X(x)}{dx^4}}}_{g(x)}$$

so $f(t) = g(x) \Rightarrow f(t) = g(x) =$ constant

call this constant $\frac{1}{\omega^2}$

$\Rightarrow$ ① $\frac{d^2 T}{dt^2} + \omega^2 T(t) = 0 \quad \Rightarrow T(t) = E \sin \omega t + F \cos \omega t.$

and

② $\frac{d^4 X}{dx^4} - \beta^4 X = 0 \quad$ where $\beta^4 = \frac{\rho A \omega^2}{EI}$

Let $X(x) = e^{\lambda x}$ substitute into ② $\Rightarrow$

$\lambda = \pm\beta, \pm i\beta$

so $X(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$

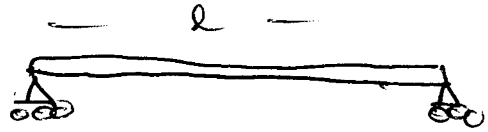
or $X(x) = A \sin \beta x + B \cos \beta x + C \sinh \beta x + D \cosh \beta x$

Unknowns: β (or ω), A, B, C, D, E, F (six unknowns)

But we have six equations.

$$
\begin{array}{rl}
 & 2 \quad \text{I.C} \\
+ & 2+2 \quad \text{B.C's.} \\
\hline
 & (6)
\end{array}
$$

Ex



Pinned-pinned beam

$$y(0,t)=0$$

$$y(l,t)=0$$

$$M(0,t) = \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0} = 0$$

$$M(l,t) = \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=l} = 0$$

$$\frac{d^2y}{dx^2} = T(t)\left[-A\beta^2 \sin\beta x - B\beta^2 \cos\beta x + C\beta^2 \sinh\beta x + D\beta^2 \cosh\beta x\right]$$

at $x=0$ $\quad y(0,t)=0 \Rightarrow B+D=0$

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0} = 0 \Rightarrow -B+D=0 \quad \Rightarrow \boxed{B=D=0}$$

at

$$y(l,t) = 0 = A\sin\beta l + C\sinh\beta l$$

$$\frac{\partial^2 y(l,t)}{\partial x^2} = 0 = -A\sin\beta l + C\sinh\beta l \quad \Rightarrow$$

$$A\sin\beta l = 0$$

$$C\sin\beta l = 0$$

But $\sinh \beta l > 0 \quad \forall \beta \cdot l > 0$

$\Rightarrow C = 0$

And $A \sin \beta l = 0 \Rightarrow$

$A = 0 \Rightarrow$ trivial condition

OR

$\sin \beta l = 0 \Rightarrow \beta_n \cdot l = n\pi \Rightarrow \beta_n = \frac{n\pi}{l}$

~~since~~

$\Rightarrow$

$$\boxed{\omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}}} \qquad \left(\text{since } \beta^4 = \frac{\rho A \omega^2}{EI}\right)$$

– natural frequencies –

and since $X(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$

we get

$$\boxed{X_n(x) = A_n \cdot \sin \beta_n x} \leftarrow n^{th} \text{ mode shape}$$

and the displacement of the beam:

$$y = \sum_{s=1}^{\infty} X_s T_s = \sum_{s=1}^{\infty} \sin\frac{n\pi}{l}\cdot x\,(E\sin\omega_s t + F\cos\omega_s t)$$

$$X_n(x) = A_n \sin\frac{n\pi}{l}x$$

E and F are determined the same way as for the bar
ie $y(0) = H(x)$, $\dot{y}(0) = G(x)$. Mult each side by $\sin\beta_r x$.