Test 2 (Key)

(1) $M = m\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, $K = k\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$

$M^{-1}K = \frac{k}{m}\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$ let $\lambda = \omega^2$, $h = \frac{k}{m}$

$$\begin{vmatrix}h-\lambda & -h & 0\\ -h & 2h-\lambda & -h\\ 0 & -h & h-\lambda\end{vmatrix} = (h-\lambda)[(2h-\lambda)(h-\lambda)-h^2] + h[(-h)(h-\lambda)] = 0$$

$(h-\lambda)[(2h-\lambda)(h-\lambda)-h^2-h^2] = 0$

$(h-\lambda)(2h^2-2h\lambda-\lambda h+\lambda^2-2h^2] = 0 \Rightarrow (h-\lambda)\,\lambda\cdot(\lambda-3h) = 0$

$\Rightarrow \omega_1 = 0$, $\omega_2 = \sqrt{\frac{k}{m}}$, $\omega_3 = \sqrt{\frac{3k}{m}}$

Eigenvectors

$$\begin{pmatrix}1-3 & -1 & 0\\ -1 & 2-3 & -1\\ 0 & -1 & 1-3\end{pmatrix}\begin{pmatrix}X_1^3\\X_2^3\\X_3^3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Let $X_1^3 = 1 \Rightarrow$ from 1st row
$-2 - X_2^3 = 0 \Rightarrow X_2^3 = -2$
$\Rightarrow 2 - 2X_3^3 = 0 \Rightarrow X_3^3 = 1$
from 3rd row

so $X^3 = \begin{pmatrix}1\\-2\\1\end{pmatrix}$

solution:

$$\underset{\sim}{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}(A_1 + B_1 t) + \begin{pmatrix}1\\0\\-1\end{pmatrix}\left(A_2\cos\sqrt{\frac{k}{m}}\,t + B_2\sin\sqrt{\frac{k}{m}}\,t\right) + \begin{pmatrix}1\\-2\\1\end{pmatrix}\left(A_3\cos\sqrt{\frac{3k}{m}}\cdot t + B_3\sin\sqrt{\frac{3k}{m}}\,t\right)$$

IC: $x(0) = \begin{pmatrix}2\\-2\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}A_1 + \begin{pmatrix}1\\0\\-1\end{pmatrix}A_2 + \begin{pmatrix}1\\-2\\1\end{pmatrix}A_3$ mult by $X^{1T}\cdot M$

$(1\ 1\ 1)\begin{pmatrix}2\\-2\\0\end{pmatrix} = (1\ 1\ 1)\begin{pmatrix}1\\1\\1\end{pmatrix}A_1 + (1\ 1\ 1)\begin{pmatrix}1\\0\\-1\end{pmatrix}A_2 + (1\ 1\ 1)\begin{pmatrix}1\\-2\\1\end{pmatrix}A_3 \Rightarrow$

$0 = 3A_1 + 0A_2 + 0A_3 \Rightarrow A_1 = 0$

Similarly: $A_2 = \dfrac{(1\ 0\ -1)\begin{pmatrix}2\\-2\\0\end{pmatrix}}{(1\ 0\ -1)\begin{pmatrix}1\\0\\-1\end{pmatrix}} = 1$, $A_3 = \dfrac{(1\ -2\ 1)\begin{pmatrix}2\\-2\\0\end{pmatrix}}{(1\ -2\ 1)\begin{pmatrix}1\\-2\\1\end{pmatrix}} = 1$

$\dot{\underset{\sim}{x}} = 0 \Rightarrow B_1 = B_2 = B_3 = 0$

$$\Rightarrow \underset{\sim}{x} = \begin{pmatrix}1\\0\\1\end{pmatrix}\cos\sqrt{\frac{k}{m}}\cdot t + \begin{pmatrix}1\\-2\\1\end{pmatrix}\cos\sqrt{\frac{3k}{m}}\cdot t \Longleftarrow \text{ans}$$

Test 2

(2) $M = m\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $k = k\begin{bmatrix}8 & -4\\ -4 & 8\end{bmatrix}$ $\qquad k = m = 1$

$M^{-1}K = \begin{pmatrix}8 & -4\\ -4 & 8\end{pmatrix}$, $\underset{\sim}{f} = \begin{bmatrix}0\\ f(t)\end{bmatrix}$

$\begin{vmatrix}8-\lambda & -4\\ -4 & 8-\lambda\end{vmatrix} = 0 \Rightarrow (8-\lambda)^2 - 16 = 0 \Rightarrow 64 - 16\lambda + \lambda^2 - 16 = 0$

$\boxed{\lambda_1 = 4 \, , \, \lambda_2 = 12}$ $\qquad$ where $\lambda = \omega^2$

Eigenvectors

(1) $\begin{pmatrix}8-4 & -4\\ -4 & 8-4\end{pmatrix}\begin{pmatrix}X_1^1\\ X_2^1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow X^1 = \begin{pmatrix}1\\ 1\end{pmatrix}$

(2) $\begin{pmatrix}8-12 & -4\\ -4 & 8-12\end{pmatrix}\begin{pmatrix}X_1^2\\ X_2^2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow X^2 = \begin{pmatrix}1\\ -1\end{pmatrix}$

so $A = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

Let $P = A x \Rightarrow \ddot{\underset{\sim}{P}} + \begin{pmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{pmatrix}\underset{\sim}{P} = (A^T M A)^{-1} A^T \cdot \underset{\sim}{f}$

$\Rightarrow \begin{pmatrix}\ddot{P}_1\\ \ddot{P}_2\end{pmatrix} + \begin{pmatrix}4 & 0\\ 0 & 12\end{pmatrix}\begin{pmatrix}P_1\\ P_2\end{pmatrix} = \left[\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right]^{-1}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\cdot\begin{pmatrix}0\\ f(t)\end{pmatrix}$

$= \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ f\end{pmatrix} = \frac{1}{2}\begin{pmatrix}f\\ -f\end{pmatrix}$

$P_1 = A_1 \cos 2t + B_1 \sin 2t + \frac{1}{2}\int_0^t \frac{1}{2} f(\tau)\sin 2(t-\tau)\, d\tau$

$P_2 = A_2 \cos\sqrt{12}\, t + B_2 \sin\sqrt{12}\, t + \frac{1}{\sqrt{12}}\int_0^t -\frac{1}{2} f(\tau)\sin\sqrt{12}(t-\tau)\, d\tau$

but $\underset{\sim}{x} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}P_1\\ P_2\end{pmatrix} \Rightarrow \underset{\sim}{x} = \begin{pmatrix}1\\ 1\end{pmatrix}\cdot P_1 + \begin{pmatrix}1\\ -1\end{pmatrix}P_2$

$x(0) = \begin{pmatrix}0\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}A_1 + \begin{pmatrix}1\\ -1\end{pmatrix}A_2 \Rightarrow A_1 = \frac{(1\ 1)\begin{pmatrix}0\\ 0\end{pmatrix}}{(1\ 1)\begin{pmatrix}1\\ 1\end{pmatrix}} = 0$ $\qquad$ also $A_2 = 0, B_1 = 0$, $B_2 = 0$

$\Rightarrow \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}\cdot\frac{2}{4}\int_0^3 \sin 2(t-\tau)\, d\tau + \begin{pmatrix}1\\ -1\end{pmatrix}\frac{1}{\sqrt{12}}\cdot\int_0^3 \sin\sqrt{12}(t-\tau)\, d\tau \Rightarrow$

for $\underline{t>3}$

② cont

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{2}\left(-\cos 2(t-\tau)\right)\left(-\frac{1}{2}\right)\Big|_0^3 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{12}}\left(-\cos\sqrt{12}(t-\tau)\right)\left(-\frac{1}{\sqrt{12}}\right)\Big|_0^3$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{4}\left[\cos 2(t-3) - \cos 2t\right] + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\left(\frac{1}{12}\cos\sqrt{12}(t-3) - \cos\sqrt{12}\,t\right)$$