5.14

EQ MOTION: $m\ddot{\xi} = EA\,\xi_{xx}$

FROM SEPARATION OF VARIABLES WE'LL HAVE

$$X_{xx} + \frac{m\omega^2}{EA} X = 0$$

B.C.: $m_1 \ddot{\xi}(0,t) = EA\,\xi_x(0)$

$$m_2 \ddot{\xi}(l,t) = -EA\,\xi_x(l)$$

USING $\xi(x,t) = \Phi(t)\,X(x)$ WITH $\Phi(t) = a_1\cos(\omega t + \phi_1)$

LETS US REWRITE THE B.C.'S AS

$$-\omega^2 m_1 X(0) = EA\,X_x(0)$$
$$-\omega^2 m_2 X(l) = -EA\,X_x(l)$$

USING $\omega^2 = \beta^2 \frac{EA}{m}$ YIELDS

$$-\beta^2 \frac{m_1}{m} X(0) = X_x(0) \qquad (1)$$
$$-\beta^2 \frac{m_2}{m} X(l) = X_x(l) \qquad (2)$$

LET $X(x) = a\cos(\beta x) + b\sin(\beta x)$ (3)

$$X_x(x) = -\beta a\sin(\beta x) + \beta b\cos(\beta x) \qquad (4)$$

USING (3) & (4) IN (1) GIVES

$$-\frac{a\beta m_1}{m} = b \qquad (5)$$

$$(3\text{-}5) \rightarrow (2) \Rightarrow -\beta^2 \frac{m_2}{m}\left[\cos(\beta l) - \frac{m_1\beta}{m}\sin(\beta l)\right] = -\beta\left[-\sin(\beta l) - \frac{m_1\beta}{m}\cos(\beta l)\right]$$

$$\boxed{\tan(\beta l) = \frac{m\beta(m_1 + m_2)}{m_1 m_2 \beta^2 - m^2}}$$

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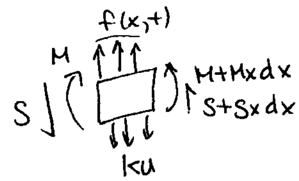

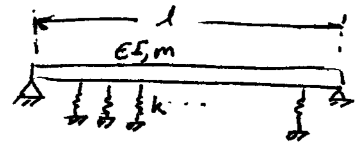

$$f\,dx + (S + S_x\,dx) - S - ku\,dx = \rho\ddot{u}\,dx$$

$$f + S_x - ku = \rho\ddot{u}$$

using $M_x = -S$ gets $\rho\ddot{u} = f - ku - M_{xx}$

using $M = EIu_{xx}$ gives $\rho\ddot{u} = f - ku - (EIu_{xx})_{xx}$

For $f = 0$ and constant ρ, EI we have

$$\rho\ddot{u} + EIu_{xxxx} + ku = 0$$

$u = X(x)\Phi(t)$ so

$$\ddot{\Phi} + \omega^2\Phi = 0$$

$$X\ddot{\Phi} + \frac{EI}{\rho}X_{xxxx}\Phi + \frac{k}{\rho}X\Phi = 0$$

$$\frac{\ddot{\Phi}}{\Phi} = -\frac{EI}{\rho}\frac{X_{xxxx}}{X} - \frac{k}{\rho} = -\omega^2 \Rightarrow X_{xxxx} - \frac{\rho}{EI}\left(\omega^2 - \frac{k}{\rho}\right)X = 0$$

If we let $\beta^4 = \frac{\rho}{EI}\left(\omega^2 - \frac{k}{\rho}\right)$ then our beam equation is in the form $X_{xxxx} - \beta^4 X = 0$ - Exactly the same as the non-foundation case. Since the BC's are pinned-pinned we have

$$X(0) = 0,\ X_{xx}(0) = 0,\ X(l) = 0,\ X_{xx}(l) = 0$$

The general solution is

$$X(x) = b_1\cos(\beta x) + b_2\sin(\beta x) + b_3\cosh(\beta x) + b_4\sinh(\beta x)$$

The boundary conditions at $x = 0$ give

$$\begin{aligned} b_1 + b_3 &= 0 \\ -b_1 + b_3 &= 0 \end{aligned} \Rightarrow b_1 = b_3 = 0$$

The solution is now $X(x) = b_2\sin(\beta x) + b_4\sinh(\beta x)$

The boundary conditions at $x = l$ give

$$b_2\sin(\beta l) + b_4\sinh(\beta l) = 0 \quad (1)$$

$$-b_2\sin(\beta l) + b_4\sinh(\beta l) = 0 \quad (2)$$

$(1)+(2) \Rightarrow b_4\sinh(\beta l) = 0 \Rightarrow b_4 = 0$

$b_2\sin(\beta l) = 0 \Rightarrow \beta l = n\pi$ so $\beta_n = \frac{n\pi}{l}$

Eigenfunctions

$$\boxed{X_n(x) = \sin\left(\frac{n\pi x}{l}\right)}$$

From $\beta^4 = \frac{\rho}{EI}\left(\omega^2 - \frac{k}{\rho}\right)$ we have

$$\omega^2 = \frac{EI\beta^4}{\rho} + \frac{k}{\rho}, \quad \boxed{\omega_n = \sqrt{\frac{EI\beta_n^4}{\rho} + \frac{k}{\rho}}}$$

5.64 THE GENERAL BEAM RESPONSE IS

$$X(x) = b_1 \cos(\beta x) + b_2 \sin(\beta x) + b_3 \cosh(\beta x) + b_4 \sinh(\beta x)$$

WHERE $\beta^4 = \frac{\omega^2 \rho}{EI}$

B.C. : $X(0) = 0$, $X_{xx}(0) = 0$, $X(l) = 0$, $X_{xx}(l) = 0$

APPLYING THE FIRST TWO B.C. :

$$\begin{aligned} b_1 + b_3 &= 0 \\ -b_1 + b_3 &= 0 \end{aligned} \Rightarrow b_1 = b_3 = 0$$

THE LAST TWO B.C. YIELD

$$b_2 \sin(\beta l) + b_4 \sinh(\beta l) = 0 \qquad (1)$$

$$-b_2 \sin(\beta l) + b_4 \sinh(\beta l) = 0 \qquad (2)$$

(1)+(2) $\Rightarrow$ $b_4 \sinh(\beta l) = 0 \Rightarrow b_4 = 0$

(1)−(2) $\Rightarrow$ $b_2 \sin(\beta l) = 0 \Rightarrow \beta_n l = n\pi, \; \beta_n = \frac{n\pi}{l}$

THUS $X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$ (SYSTEM EIGENFUNCTION)

NOW EXPRESS THE FORCING IN TERMS OF THE SYSTEM EIGENFUNCTIONS:

$$\bar{f}\, \delta\left(x - \frac{l}{2}\right) = \sum_{n=1}^{\infty} \bar{f}_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\int_0^l \bar{f}\, \delta\left(x - \frac{l}{2}\right) \sin\left(\frac{m\pi x}{l}\right) dx = \sum_{n=1}^{\infty} \bar{f}_n \int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx$$

$$\bar{f} \sin\left(\frac{m\pi}{2}\right) = \bar{f}_n \frac{l}{2} \Rightarrow \bar{f}_n = \frac{2}{l} \bar{f} \sin\left(\frac{m\pi}{2}\right)$$

$$\bar{f}_1 = \frac{2\bar{f}}{l}, \quad \bar{f}_2 = 0, \quad \bar{f}_3 = -\frac{2\bar{f}}{l}, \quad \bar{f}_4 = 0, \quad \bar{f}_5 = \frac{2\bar{f}}{l} \; \ldots$$

$$\bar{f}_{(2n-1)} = \frac{2\bar{f}}{l}(-1)^{n+1} \quad n = 1,2,3,\ldots \; ; \quad \bar{f}_{2n} = 0 \quad n = 1,2,3\ldots$$

5.64 (cont)

FROM SECTION 5.4:

$$\ddot{a}_m(t)\rho\int_0^l \sin^2\left(\frac{m\pi x}{l}\right)dx = -a_m(t)EI\left(\frac{m\pi}{l}\right)^4\int_0^l \sin^2\left(\frac{m\pi x}{l}\right)dx$$
$$+\bar{f}_m\cos(\omega_f t)\int_0^l \sin^2\left(\frac{m\pi x}{l}\right)dx$$

$$\ddot{a}_m + \frac{EI(m\pi)^4}{\rho l^4}a_m = \frac{\bar{f}_m}{\rho}\cos(\omega_f t)$$: EQS OF MOTION

SOLUTION: $$a_m(t) = \frac{\bar{f}_m}{\rho}\frac{\cos(\omega_f t)}{\omega_m^2-\omega_f^2} \quad ; \quad \omega_m^2 = \frac{EI(m\pi)^4}{\rho l^4}$$

$$\boxed{u(x,t) = \sum_{m=1}^{\infty}\frac{\bar{f}_m\sin\left(\frac{m\pi x}{l}\right)\cos(\omega_f t)}{\rho(\omega_m^2-\omega_f^2)}}$$