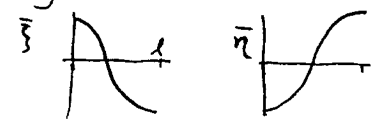
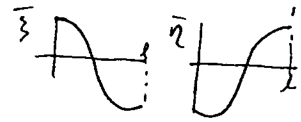
5.44 (CONT) The k_1 spring isn't deformed during the motion



mode shapes

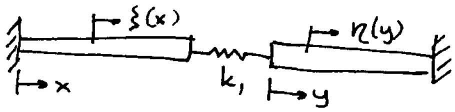
The next mode will look like



$\beta_3 = 3.157427$ (higher than π due to the k_1 participation)

$\omega_3 = 25{,}785$ rad/s

5.45



eigen functions: $\bar{\xi}(x) = a_1 \cos\beta x + b_1 \sin\beta x$, $\bar{\eta}(y) = a_2 \cos\beta y + b_2 \sin\beta y$

Since the beams are identical the same β's are used for each beam

B.C.: $\bar{\xi}(0) = 0 \Rightarrow a_1 = 0$ $\quad (\bar{\xi}(x) = b_1 \sin\beta x)$

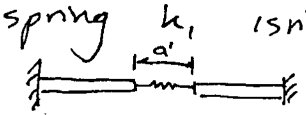
B.C.: $\bar{\eta}(l) = 0 \Rightarrow a_2 \cos\beta l + b_2 \sin\beta l = 0$

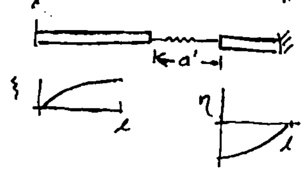
B.C.: $[\bar{\eta}(0) - \bar{\xi}(l)]k_1 - EA\,\bar{\xi}_x(l) = 0 \Rightarrow (a_2 - b_1 \sin\beta l)k_1 - EA b_1 \beta \cos\beta l = 0$

B.C. $[\bar{\xi}(l) - \bar{\eta}(0)]k_1 + EA\,\bar{\eta}_y(0) = 0 \Rightarrow (b_1 \sin\beta l - a_2)k_1 + EA[b_2 \beta]$

$$\begin{bmatrix} \cos\beta l & 0 & \sin\beta l \\ k_1 & -k_1 \sin\beta l - EA\beta\cos\beta l & 0 \\ -k_1 & k_1 \sin\beta l & EA\beta \end{bmatrix} \begin{Bmatrix} a_2 \\ b_1 \\ b_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

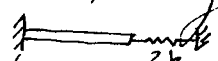
The first β is $\beta_1 = \frac{\pi}{2}$. This corresponds to each bar undergoing fixed-free vibrations. Both ends move in phase and thus the spring k_1 isn't deformed





mode shapes

For β_2 we have symmetric motions towards & away from each other. The β is equivalent to the one found for the illustrated "half" system ↑



$\beta_2 = 1.602$ (higher than β_1 since the spring is now involved)

5.51

For the clamped-clamped case we have

$\bar{W}_x(0) = \bar{W}(0) = 0$ & $\bar{W}_x(l) = \bar{W}(l) = 0$. We know $\bar{W}(x) = a\cos\beta x + b\sin\beta x + c\cosh\beta x + d\sinh\beta x$

$\bar{W}(0) = 0 \Rightarrow a + c = 0 \Rightarrow a = -c$

$\bar{W}_x(0) = 0 \Rightarrow b + d = 0 \Rightarrow b = -d$

so $\bar{W}(x) = a(\cos\beta x - \cosh\beta x) + b(\sin\beta x - \sinh\beta x)$

$\bar{W}(l) = 0 \Rightarrow a(\cos\beta l - \cosh\beta l) + b(\sin\beta l - \sinh\beta l) = 0$

$\bar{W}_x(l) = 0 \Rightarrow a(-\sin\beta l - \sinh\beta l) + b(\cos\beta l - \cosh\beta l) = 0$

$$\begin{bmatrix} \cos\beta l - \cosh\beta l & \sin\beta l - \sinh\beta l \\ -\sin\beta l - \sinh\beta l & \cos\beta l - \cosh\beta l \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Taking the determinant: $(\cos\beta l - \cosh\beta l)(\cos\beta l - \cosh\beta l) + (\sin\beta l + \sinh\beta l)(\sin\beta l - \sinh\beta l) = 0$

$\cos^2\beta l + \sin^2\beta l - 2\cos\beta l\cosh\beta l + \cosh^2\beta l - \sinh^2\beta l = 0$

$2 - 2\cos\beta l\cosh\beta l = 0$, $\boxed{\cos\beta l\cosh\beta l = 1}$

For the free-free case we have $\bar{W}_{xx}(0) = \bar{W}_{xxx}(0) = \bar{W}_{xx}(l) = \bar{W}_{xxx}(l) = 0$

$\bar{W}_{xxx}(0) = 0 \Rightarrow -b + d = 0$ or $b = d$

$W_{xx}(0) = 0 \Rightarrow -a + c = 0$ or $a = c$

$$\bar{W}(x) = a(\cos\beta x + \cosh\beta x) + b(\sin\beta x + \sinh\beta x)$$

$\bar{W}_{xx}(l) = 0 \Rightarrow a(-\cos\beta l + \cosh\beta l) + b(-\sin\beta l + \sinh\beta l) = 0$

$\bar{W}_{xxx}(l) = 0 \Rightarrow a(\sin\beta l + \sinh\beta l) + b(-\cos\beta l + \cosh\beta l) = 0$

$$\begin{bmatrix} -\cos\beta l + \cosh\beta l & -\sin\beta l + \sinh\beta l \\ \sin\beta l + \sinh\beta l & -\cos\beta l + \cosh\beta l \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

determinant: $(-\cos\beta l + \cosh\beta l)^2 - (\sin\beta l + \sinh\beta l)(-\sin\beta l + \sinh\beta l) = 0$

$\cos^2\beta l - 2\cos\beta l\cosh\beta l + \cosh^2\beta l + \sin^2\beta l - \sinh^2\beta l = 0$

$2 - 2\cos\beta l\cosh\beta l = 0$

$\boxed{\cos\beta l\cosh\beta l = 1}$

Both cases have the same β equation and therefore the same β_n's.

5.67

$$T y_{xx} - \rho \ddot{y} = 0, \quad y(0)=0, \quad y_x(l)=0$$

ASSUME $y = \bar{y}\sin(\omega t)$

$$T\bar{y}_{xx} + \rho\omega^2 \bar{y} = 0$$

$$\bar{y}_{xx} + \frac{\rho\omega^2}{T}\bar{y} = 0$$

$$\bar{y}_{xx} + \beta^2 \bar{y} = 0 \quad , \quad \beta^2 = \frac{\rho\omega^2}{T}$$

SOLUTION: $\bar{y}(x) = a_1\cos(\beta x) + a_2 \sin(\beta x)$

APPLYING $\bar{y}(0) = 0$ GIVES $a_1 = 0$

$\bar{y}(x) = a_2 \sin(\beta x)$

APPLYING $\bar{y}_x(l) = 0$ GIVES

$$\cos(\beta l) = 0 \Rightarrow \beta_n l = \frac{(2n-1)\pi}{2} \qquad n = 1, 2 \cdots$$

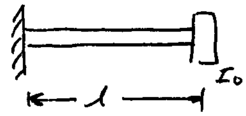
$$\beta_n = \frac{(2n-1)\pi}{2l}$$

EIGENFUNCTION IS $\sin\left(\frac{(2n-1)\pi x}{l}\right)$

$$\int_0^l \sin\left(\frac{(2n-1)\pi x}{l}\right)\sin\left(\frac{(2m-1)\pi x}{l}\right)dx = 0 \qquad n \neq m$$

EIGENFUNCTION SATISFIES ORTHOGONALITY

5.40



$r_i = .01$ $\quad l = 2$
$r_o = .011$ $\quad G = 5.7\times10^{10}$
$\rho = 6.7\times10^{3}$

$\bar\theta(x) = a_1 \cos\beta x + a_2 \sin\beta x$

$\bar\theta(0) = 0 \Rightarrow a_1 = 0$

AT $x = l$ WE HAVE THE B.C. $I_0\ddot\theta = -GJ\theta_x$

$-\omega^2 I_0 \bar\theta(l) = -GJ\bar\theta_x(l)$

$\omega^2 = \beta^2\left(\frac{G}{\rho}\right)$ SO THIS BECOMES $-\beta^2\frac{I_0 G}{\rho}\bar\theta(l) + GJ\bar\theta_x(l) = 0$

$-\beta^2\frac{I_0}{\rho}\sin\beta l + J\beta\cos\beta l = 0$

$-\beta I_0 \sin\beta l + J\rho\cos\beta l = 0 \qquad (1)$

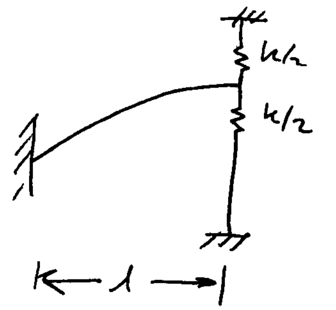
ADDING I_0 TO THE SHAFT REDUCES THE FIXED-FREE NATURAL FREQUENCIES. FOR THE FIXED-FREE CASE THE BOUNARY CONDITION $\bar\theta_x(l) = 0$ GIVES β SOLUTIONS OF $\beta_n = \left(\frac{2n-1}{2}\right)\frac{\pi}{l}$. THE FIRST NATURAL FREQUENCY β IS THUS $\beta_1 = \frac{\pi}{2l}$. THE PROBLEM IS THEREFORE TO FIND AN I_0 SUCH THAT THE NEW β_1 IS 90% OF THE ORIGINAL, OR $0.9\left(\frac{\pi}{2l}\right) = 0.9\left(\frac{\pi}{(2(2))}\right) = .7069$

CALCULATE J: $J = \frac{\pi}{2}(.011^4 - .01^4) = 7.29\times10^{-9}$

$(1) \Rightarrow -(.7069)I_0\sin(.7069(2)) + (7.29\times10^{-9})(6.7\times10^{3})\cos(.7069(2)) = 0$

$$\boxed{I_0 = 1.09\times10^{-5}\ \text{kg}\cdot\text{m}^2}$$

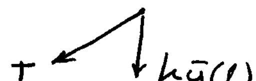
5.16



$T = 100, \ \rho = .002$

$x = 0: \ \boxed{\bar{y}(0) = 0}$

$x = l:$



$$\boxed{k\bar{y}(l) - T\bar{y}_x(l) = 0}$$

$$\bar{y}(x) = a_1 \sin\beta x + a_2 \cos\beta x$$

$$\bar{y}(0) = 0 \Rightarrow a_2 = 0$$

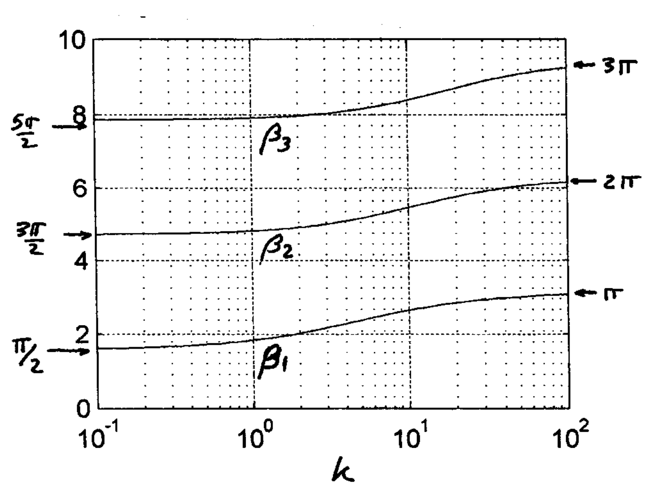
B.C. @ $x = l$: $\quad k a_1 \sin\beta l + T\beta a_1 \cos\beta l = 0$

$$\boxed{k \sin\beta l = -T\beta \cos\beta l}$$

WITH $l = 1$, $T = 100$ THIS IS

$$k \sin\beta + 100\beta \cos\beta = 0$$

THE FIRST THREE β'S, PLOTTED AS A FUNCTION OF k, ARE SHOWN BELOW



5.18 The general solution for a tensioned string problem is found from the equation of motion:

$$T y_{xx} - \rho \ddot{y} = 0$$

If $y(x,t) = \bar{y}(x)\cos(\omega t + \phi)$ we have

$$T\bar{y}_{xx} + \omega^2 \rho \bar{y} = 0 \ , \quad \bar{y}_{xx} + \frac{\omega^2 \rho}{T}\bar{y} = 0$$

Set $\beta^2 = \omega^2 \frac{\rho}{T}$ to obtain $\bar{y}_{xx} + \beta^2 \bar{y} = 0$

general solution to this is $\bar{y}(x) = a\sin\beta x + b\cos\beta x$

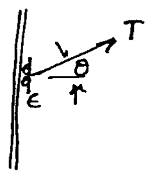
For the fully restrained case $\bar{y}(0) = \bar{y}(l) = 0$

$\bar{y}(0) = 0 \Rightarrow b = 0$ so $\bar{y}(x) = a\sin\beta x$

$\bar{y}(l) = 0 \Rightarrow a\sin\beta l = 0 \Rightarrow \beta_n = \frac{n\pi}{l}$

For the unrestrained case we have $\bar{y}_x(0) = \bar{y}_x(l) = 0$

This can be seen from a force balance:



Assume for a moment that there is a mass ϵ at $x=0$. We have

$\epsilon \ddot{y}\big|_{x=0} = T\sin\theta$. Since $\sin\theta \approx \frac{dy}{dx}\big|_0$ we have

$\epsilon \ddot{y}\big|_{x=0} = T y_x\big|_{x=0}$. But actually there is no mass ϵ so we have $0 = T y_x\big|_{x=0}$. This can only be satisfied if $y_x\big|_{x=0} = 0$. The same holds at $x = l$

$\bar{y}_x\big|_{x=0} = 0 \Rightarrow \beta(a\cos\beta\cdot 0 - b\sin\beta\cdot 0) = 0 \Rightarrow a = 0$

Thus $\bar{y}(x) = b\cos\beta x$

$y_x\big|_{x=l} = 0 \Rightarrow -b\beta\sin\beta l = 0 \Rightarrow \beta_n = \frac{n\pi}{l}$

Thus we see that the β's for the two cases are the same but the eigenfunctions are different ($\sin\beta_n x$ for restrained and $\cos\beta_n x$ for unrestrained)