3.13 Equations of motion: $5\ddot{x}+20x = 5\sum_{n=1}^{\infty}\frac{8a}{(n\pi)^2}\sin\frac{n\pi}{2}\sin\frac{2n\pi t}{T}$

Solve $\quad 5\ddot{x}_n + 20x_n = 5\frac{8a}{(n\pi)^2}\sin\frac{n\pi}{2}\sin\frac{2n\pi t}{T}$

$$x_n = a_n \sin\frac{2n\pi t}{T}$$

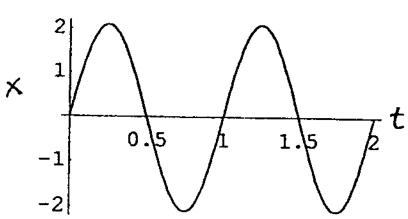
$$\ddot{x}_n = -a_n\left(\frac{2n\pi}{T}\right)^2\sin\frac{2n\pi t}{T}$$

$$-5a_n\left[\frac{2n\pi}{T}\right]^2\sin\frac{2n\pi t}{T} + 20a_n\sin\frac{2n\pi t}{T} = \frac{40a}{(n\pi)^2}\sin\frac{n\pi}{2}\sin\frac{2n\pi t}{T}$$

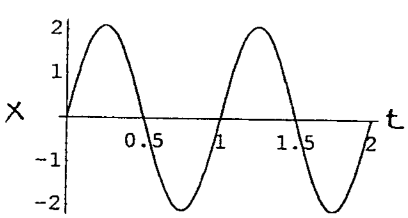
$$a_n = \frac{\frac{40a}{(n\pi)^2}\sin\frac{n\pi}{2}}{20-5\left(\frac{2n\pi}{T}\right)^2}$$

$$x_n = \frac{\frac{8a}{(n\pi)^2}\sin\frac{n\pi}{2}}{4-\left(\frac{2n\pi}{T}\right)^2}\sin\frac{2n\pi t}{T} \quad ; \quad x = \sum_{n=1}^{\infty} x_n$$

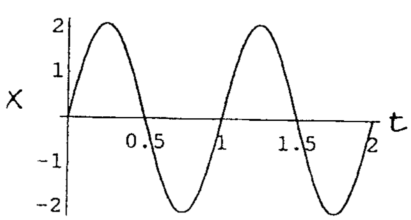
If we use $T=3$ and plot the response for 1, 2 and 3 retained terms we'll obtain the following:



1 term



2 terms

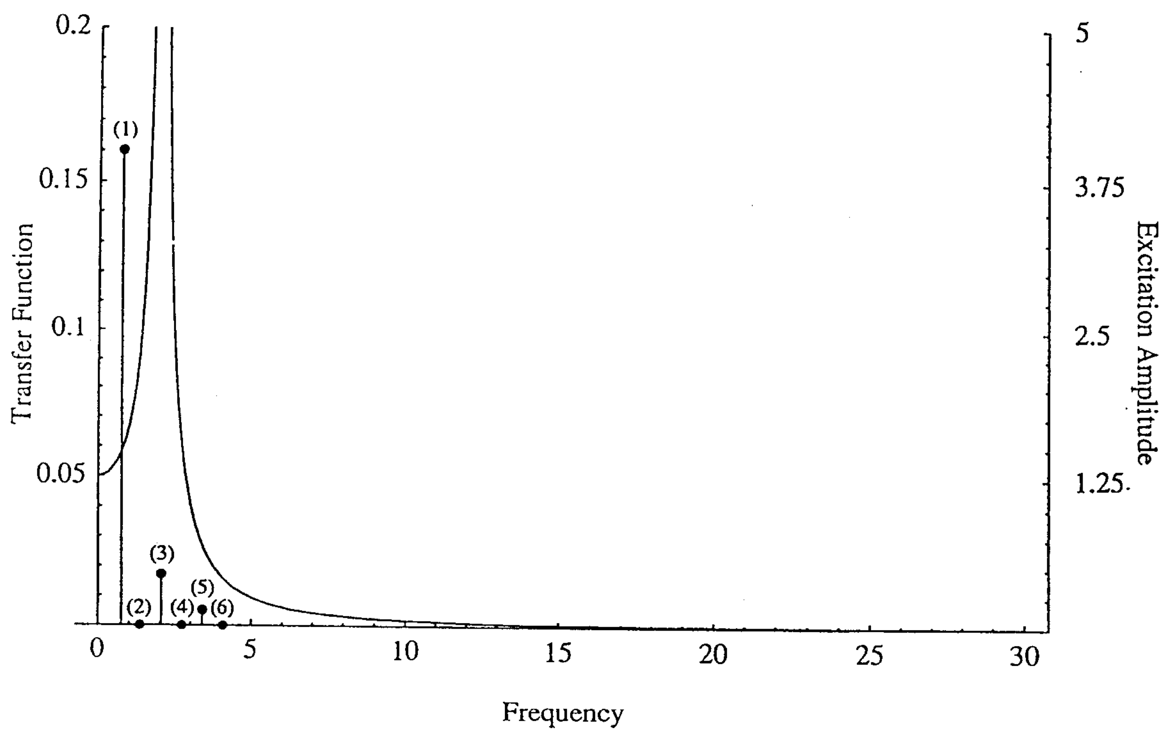


3 terms

Note that they all look the same. At least to graphical precision, the additional terms didn't affect the result. Now let's change T from 3 seconds to 9.24 seconds:

For $T = 9.24$ the situation is quite different. The 1st term (1) is still much greater than the 2nd (labeled (3) since it corresponds to $3\omega_0$) But (3) is now almost at resonance. Thus it is greatly amplified. For this case the 2nd term is dominant. The first isn't negligible and causes the observed lower frequency oscillation.

$T = 9.24\,s$



3.18

$$h(t) = \frac{1}{m\omega_n} \sin(\omega_n t)$$

$$x(t) = \int_0^t \frac{1}{m\omega_n} \sin(\omega_n \tau)\left[\delta(t-\tau) + \delta(t-\tau-.4) + \delta(t-\tau-.8) + \cdots\right] d\tau$$

$$= \frac{1}{m\omega_n}\left(\sin(\omega_n t) + \sin(\omega_n(t-.4)) + \sin(\omega_n(t-.8)) + \cdots\right)$$

$$\omega_n = \sqrt{\frac{2500\pi^2}{100}} = 5\pi \quad \text{so}$$

$$x(t) = \frac{1}{5\pi m}\left(\sin(5\pi t) + \sin(5\pi t - 2\pi) + \sin(5\pi t - 4\pi) + \cdots\right)$$

$$= \frac{n}{5\pi m}\left(\sin(5\pi t)\right)$$ WHERE n IS THE NUMBER OF DIRAC DELTA FUNCTIONS THAT HAVE BEEN APPLIED.

AS $t \to \infty$, $n \to \infty$ AND THEREFORE THE OSCILLATIONS GROW WITHOUT BOUND – THE MOTION IS NOT STABLE

3.19

$$\left|\frac{\ddot{x}}{g}\right|_{max} = \sqrt{1 + \left(\frac{v\omega_n}{g}\right)^2} \qquad v = \sqrt{2gh} = 4.43 \qquad \omega_n = \sqrt{\frac{20000}{20}} = 31.62$$

$$\boxed{|\ddot{x}|_{max} = 9.81\sqrt{1 + \left(\frac{(4.43)(31.62)}{9.81}\right)^2} = 140.4 \text{ m/s}^2}$$

3.20

35 mph $= 51.\bar{3}$ ft/s

$$a = \frac{v(0.1) - v(0)}{\Delta t} = \frac{0 - 51.\bar{3}}{0.1} = -51\bar{3}.\bar{3} \text{ ft/sec}^2$$

$$\boxed{a = -\frac{513.\bar{3}}{32.2} = 16g}$$

3.23 INITIAL SYSTEM : $k_o = 400 N/m$, $m = 0.05 kg$

$$\omega_n = \sqrt{\frac{400}{0.05}} = 89.4 \text{ rad/s}$$

$$|\ddot{x}|_{max} = g\sqrt{1+\left(\frac{v\omega_n}{g}\right)^2}$$

FOR A 2m DROP WE HAVE

$$v = \sqrt{2gh} = \sqrt{2(9.81)(2)} = 6.26 \text{ m/s}$$

$$|\ddot{x}|_{max} = 9.81\sqrt{1+\left(\frac{(6.26)(89.4)}{9.81}\right)^2} = 560 \text{ m/s}^2$$

THUS 560 m/s^2 IS THE EGG BREAKING LIMIT. TO RAISE THE DROP HEIGHT TO 4 m WE NEED TO SOLVE

$$560 = 9.81\sqrt{1+\frac{v^2k}{g^2m}}\,.$$

FROM 4 m WE HAVE $v = \sqrt{2(9.81)(4)} = 8.86 \text{ m/s}$

THUS $$560 = 9.81\sqrt{1+\frac{(8.86)^2k}{(9.81)^2(0.05)}} \Rightarrow \boxed{k = 200 N/m}$$

REDUCING k IN HALF ALLOWS THE DROP HEIGHT TO BE DOUBLED

3.24 FROM $|\ddot{x}|_{max} = g\sqrt{1+\left(\frac{v}{g}\right)^2\frac{k}{m}}$ WE SEE THAT IF THE SURFACE STIFFNESS GOES TO INFINITY (RIGID SURFACE) THEN $|\ddot{x}|_{max}$ ALSO GOES TO INFINITY. THIS REFLECTS THE FACT THAT THE VELOCITY GOES FROM V_{IMPACT} TO ZERO IN ZERO SECONDS, i.e. INFINITE NEGATIVE ACCELERATION

4.2

$$m_1\ddot{x}_1 = -k_1 x_1 + (x_2 - x_1)k_2$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - x_3 k_3$$
$$m_3(\ddot{x}_2 - \ddot{x}_3) = k_3 x_3$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & m_3 & -m_3 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2 & k_3 \\ 0 & 0 & -k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

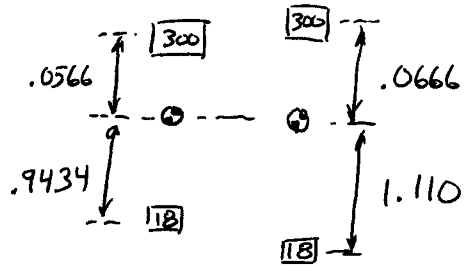
$$\begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & .02 & -.02 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 2500 & -1500 & 0 \\ -1500 & 1500 & 200 \\ 0 & 0 & -200 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

USING MATLAB

$$\boxed{\begin{aligned} \omega_1 &= 28.5 \text{ RAD/S}, \quad \mathbf{X}_1 = \begin{Bmatrix} -.5810 \\ -.8107 \\ .0719 \end{Bmatrix} \\ \omega_2 &= 83.0 \text{ RAD/S}, \quad \mathbf{X}_2 = \begin{Bmatrix} .5480 \\ -.3446 \\ .7622 \end{Bmatrix} \\ \omega_3 &= 103.4 \text{ RAD/S}, \quad \mathbf{X}_3 = \begin{Bmatrix} .0343 \\ -.0652 \\ -.9973 \end{Bmatrix} \end{aligned}}$$

(4.3 CONT)

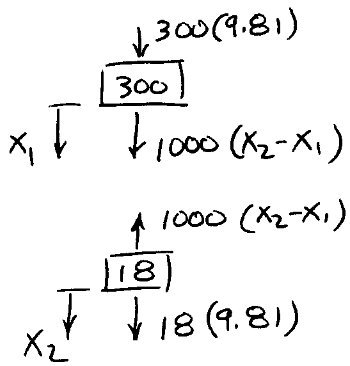
LINE UP THE CENTER OF MASS FOR THE 2 CASES



$.0666 - .0566 = .01$

$1.110 - .9434 = .1666$

TO ANALYZE THE SYSTEM RESPONSE WE'LL REFERENCE OUR COORDINATES TO THE PLACES THAT THE LEDGE AND DOPEY WOULD OCCUPY IN THE ABSENCE OF GRAVITY, KEEPING THE CENTER OF MASS FIXED AT .0666 m BELOW THE UNBROKEN LEDGE. THUS, THE SYSTEM INITIAL CONDITIONS ARE THE DISPLACEMENTS NEEDED TO GET DOPEY AND THE LEDGE TO THEIR PRE-BREAK POSITIONS



$$300\ddot{x}_1 = 1000(x_2 - x_1) + 300g$$
$$18\ddot{x}_2 = -1000(x_2 - x_1) + 18g$$

$$\ddot{x}_1 + 3.\bar{3}x_1 - 3.\bar{3}x_2 = 9.81$$
$$\ddot{x}_2 + 55.\bar{5}x_2 - 55.\bar{5}x_1 = 9.81$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3.\bar{3} & -3.\bar{3} \\ -55.\bar{5} & 55.\bar{5} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} 9.81$$

FREE VIBRATION:

$$\begin{vmatrix} 3.\bar{3} - \omega^2 & -3.\bar{3} \\ -55.\bar{5} & 55.\bar{5} - \omega^2 \end{vmatrix} = 0 \Rightarrow \omega^4 - 58.\bar{8}\omega^2 = 0$$

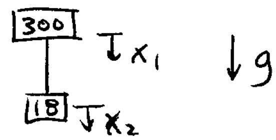
$$\omega^2(\omega^2 - 58.\bar{8}) = 0$$

$\omega_1 = 0$ CORRESPONDS TO TRANSLATION OF MASS CENTER

$\omega_2 = \sqrt{58.8} = 7.67$ RAD/S CORRESPONDS TO OSCILLATION OF THE FALLING DOPEY/LEDGE SYSTEM

4.5 IF THE CONNECTION IS AN INEXTENSIBLE STRING THEN AFTER THE BREAK BOTH THE LEDGE AND DOPEY WILL FALL AT $9.81 \, m/s^2$.

$$\ddot{x}_1 = \ddot{x}_2 = 9.81$$



FROM 4.3 WE HAVE

$$x_1(t) = -0.01\cos(7.67t) + 4.905t^2$$
$$x_2(t) = 0.1666\cos(7.67t) + 4.905t^2$$

THE AMPLITUDES (0.1666 AND 0.01) CAME ABOUT FROM THE STRETCH OF THE BUNGIE CORD DUE TO GRAVITY BEFORE THE BREAK. AS k INCREASES, BOTH AMPLITUDES GO TO ZERO. THE FREQUENCY OF 7.67 RAD/S IS DUE TO THE SPRING FORCE IN THE BUNGIE CORD. AS $k \to \infty$, THIS FREQUENCY LIKEWISE GOES TO INFINITY. IN THE LIMIT WE HAVE ULTRA HIGH FREQUENCY OSCILLATION AT VANISHINGLY SMALL AMPLITUDES. THE LIMIT CASE IS THE STRING — NO OSCILLATIONS

4.8
$$m_1\ddot{x}_1 = -k_1x_1 + k_3(x_3 - x_1) + k_2(x_2 - x_1)$$
$$m_2\ddot{x}_2 = k_4(x_3 - x_2) - k_2(x_2 - x_1)$$
$$m_3\ddot{x}_3 = -k_5X_3 - k_3(x_3 - x_1) - k_4(x_3 - x_2)$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_2+k_3 & -k_2 & -k_3 \\ -k_2 & k_2+k_4 & -k_4 \\ -k_3 & -k_4 & k_3+k_4+k_5 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & .02 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 3020 & -20 & -2000 \\ -20 & 40 & -20 \\ -2000 & -20 & 3020 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

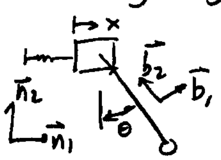
$\omega_1 = 22.3$ RAD/S, $\quad \underline{X}_1 = \begin{Bmatrix} .515 \\ .685 \\ .515 \end{Bmatrix}$

$\omega_2 = 44.9$ RAD/S, $\quad \underline{X}_2 = \begin{Bmatrix} .007 \\ -1.00 \\ .007 \end{Bmatrix}$

$\omega_3 = 50.1$ RAD/S, $\quad \underline{X}_3 = \begin{Bmatrix} .707 \\ 0 \\ -.707 \end{Bmatrix}$

$\underline{X}_2$ AND $\underline{X}_3$ SHOW STRONG DECOUPLING – THE SYSTEM COULD BE VIEWED AS TWO SEPARATE SUB-SYSTEMS. IN $\underline{X}_2$ ONLY THE INNER MASS IS MOVING APPRECIABLY, THE OUTER ONES ARE RELATIVELY STATIONARY. IN $\underline{X}_3$ ONLY THE OUTER MASSES ARE MOVING – THE INNER MASS IS STATIONARY

4.11 Using Lagrange's Eqs:



$$\vec{V}_{m_1} = \dot{x}\vec{n}_1$$

$$\vec{V}_{m_2} = \dot{x}\vec{n}_1 + l\dot{\theta}\vec{b}_1 = \dot{x}\vec{n}_1 + l\dot{\theta}(\vec{n}_1\cos\theta + \vec{n}_2\sin\theta)$$
$$= \vec{n}_1(\dot{x} + l\dot{\theta}\cos\theta) + \vec{n}_2(l\dot{\theta}\sin\theta)$$

$$KE = \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2[(\dot{x}+l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2]$$
$$= \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2(\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2)$$

$$KE = \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + \tfrac{1}{2}m_2 l^2\dot{\theta}^2 + lm_2\dot{x}\dot{\theta}\cos\theta$$

$$PE = lm_2 g(1-\cos\theta) + \tfrac{1}{2}k_1 x^2$$

$L = KE - PE$. Using $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow$

$$(m_1+m_2)\ddot{x} + \frac{d}{dt}(lm_2\dot{\theta}\cos\theta) + k_1 x = 0$$

$$(m_1+m_2)\ddot{x} + lm_2\ddot{\theta}\cos\theta - lm_2\dot{\theta}^2\sin\theta + k_1 x = 0 \quad (1) \text{ and}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow m_2 l^2\ddot{\theta} + \frac{d}{dt}(lm_2\dot{x}\cos\theta) - \frac{\partial}{\partial\theta}(lm_2\dot{x}\dot{\theta}\cos\theta) + m_2 g l\sin\theta = 0$$

$$m_2 l^2\ddot{\theta} + lm_2\ddot{x}\cos\theta - \cancel{lm_2\dot{x}\dot{\theta}\sin\theta} + \cancel{lm_2\dot{x}\dot{\theta}\sin\theta} + m_2 g l\sin\theta = 0 \quad (2)$$

Linearizing 1 & 2 (small x & θ) gives

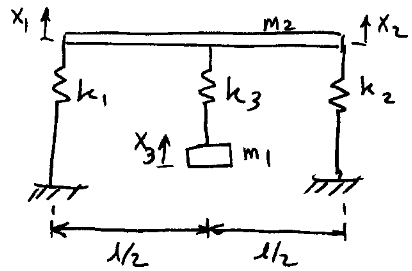
$$\boxed{\begin{aligned}(m_1+m_2)\ddot{x} + lm_2\ddot{\theta} + k_1 x = 0\\ m_2 l^2\ddot{\theta} + lm_2\ddot{x} + m_2 g l\theta = 0\end{aligned}}$$

FOR THE GIVEN PARAMETERS WE HAVE

$$\begin{bmatrix}32 & 4\\ 4 & 8\end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\end{Bmatrix} + \begin{bmatrix}15 & 0\\ 0 & 39.24\end{bmatrix}\begin{Bmatrix}x\\ \theta\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \Rightarrow \omega_1 = 0.6824 \text{ RAD/S}, \begin{Bmatrix}X\\ \Theta\end{Bmatrix}_1 = \begin{Bmatrix}0.9986\\ 0.0524\end{Bmatrix}$$

$$\omega_2 = 2.2948 \text{ RAD/S}, \begin{Bmatrix}X\\ \Theta\end{Bmatrix}_2 = \begin{Bmatrix}-0.1359\\ 0.9907\end{Bmatrix}$$

4.14



$$KE = \frac{1}{2} m_2 \left(\frac{\dot{x}_1 + \dot{x}_2}{2} \right)^2 + \frac{1}{2} \frac{m_2 l^2}{12} \left(\frac{\dot{x}_2 - \dot{x}_1}{l} \right)^2 + \frac{1}{2} m_1 \dot{x}_3^2$$

$$PE = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_3 \left(\frac{x_1 + x_2}{2} - x_3 \right)^2$$

$$L = KE - PE$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0$$

$$\left(\frac{m_2}{2} + \frac{m_2}{12} \right) \ddot{x}_1 + \frac{5 m_2}{12} \ddot{x}_2 + \left(k_1 + \frac{k_3}{4} \right) x_1 + \frac{k_3}{4} x_2 - \frac{k_3}{2} x_3 = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0$$

$$\frac{5 m_2}{12} \ddot{x}_1 + \left(\frac{m_2}{2} + \frac{m_2}{12} \right) \ddot{x}_2 + \frac{k_3}{4} x_1 + \left(k_2 + \frac{k_3}{4} \right) x_2 - \frac{k_3}{2} x_3 = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_3} - \frac{\partial L}{\partial x_3} = 0$$

$$m_1 \ddot{x}_3 - \frac{k_3}{2} x_1 - \frac{k_3}{2} x_2 + k_3 x_3 = 0$$

$$\begin{bmatrix} \frac{7 m_2}{12} & \frac{5 m_2}{12} & 0 \\ \frac{5 m_2}{12} & \frac{7 m_2}{12} & 0 \\ 0 & 0 & m_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + \frac{k_3}{4} & \frac{k_3}{4} & -\frac{k_3}{2} \\ \frac{k_3}{4} & k_2 + \frac{k_3}{4} & -\frac{k_3}{2} \\ -\frac{k_3}{2} & -\frac{k_3}{2} & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 875 & 625 & 0 \\ 625 & 875 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 27{,}500 & 17{,}500 & -35{,}000 \\ 17{,}500 & 29{,}500 & -35{,}000 \\ -35{,}000 & -35{,}000 & 70{,}000 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\boxed{\omega_1 = 2.6504 \text{ rad/s}, \quad X_1 = \begin{Bmatrix} .6352 \\ .5112 \\ .5790 \end{Bmatrix}; \quad \omega_2 = 6.6385 \text{ rad/s}, \quad X_2 = \begin{Bmatrix} .6946 \\ -.7192 \\ -.0131 \end{Bmatrix}; \quad \omega_3 = 26.8993 \text{ rad/s}, \quad X_3 = \begin{Bmatrix} .0334 \\ .0338 \\ -.9989 \end{Bmatrix}}$$

4.16 For $[K]X_i = 0$ we'd need $\omega_i = 0$ since $[M]\ddot{X} + [K]X = 0$ & so $\omega_i^2[M]X_i = [K]X_i$. This is possible if the system has a rigid body, or zero frequency mode (associated with uniform velocity as opposed to sinusoidal oscillations). As we've seen in the text, $[K]$ would have to have a zero determinant, i.e. be noninvertible. If it was invertible then

$$[K]X_i = 0$$
$$[K]^{-1}[K]X_i = [K]^{-1}0 = 0$$
$$X_i = 0.$$

Since $X_i \neq 0$ (it's an eigenvector), we conclude that $[K]$ isn't invertible.

4.17 $X_1 = \begin{Bmatrix} 4 \\ 2 \\ 2 \end{Bmatrix}$, $X_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$, $X_3 = \begin{Bmatrix} 3 \\ 2 \\ 3 \end{Bmatrix}$

$X_3 = X_1 - X_2$ THUS THE THREE VECTORS AREN'T INDEPENDENT. ANOTHER WAY TO SEE THIS IS TO FIND THE DETERMINANT OF $[X_1 \vdots X_2 \vdots X_3]$

$\left| [X_1 \vdots X_2 \vdots X_3] \right| = 0$, WHICH TELLS US THAT THE COLUMNS ARE LINEARLY DEPENDENT. BECAUSE THEY AREN'T INDEPENDENT, THE VECTORS CAN'T BE EIGENVECTORS

4.21

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

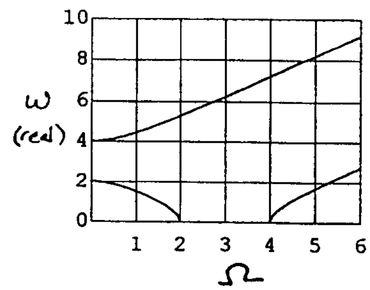
$$\begin{vmatrix} 2-\omega^2 & -1 \\ -1 & 3-\omega^2 \end{vmatrix} = 0$$

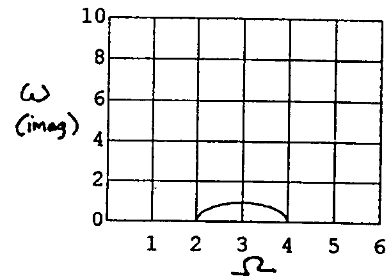
$$\omega_1 = 1.1756 \text{ RAD/S}, \quad X_1 = \begin{Bmatrix} .8507 \\ .5257 \end{Bmatrix}$$

$$\omega_2 = 1.9021 \text{ RAD/S}, \quad X_2 = \begin{Bmatrix} -.5257 \\ .8507 \end{Bmatrix}$$

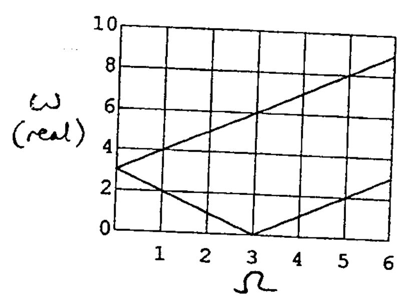
$$\boxed{\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = a_1 \begin{Bmatrix} .8507 \\ .5257 \end{Bmatrix} \cos(1.1756t) + a_2 \begin{Bmatrix} -.5257 \\ .8507 \end{Bmatrix} \sin(1.9021t)}$$

4.27 (CONT) This gives us $\Omega^4 - 20\Omega^2 + 64 = 0$ (after simplifying)
or $(\Omega^2-4)(\Omega^2-16)=0 \Rightarrow \Omega = 2, 4$ are critical
The values for ω are shown below for $0 \le \Omega \le 6$





Note that from $0 \le \Omega < 2$ & $4 \le \Omega \le 6$ we have two real roots. These represent the oscillations you'd see in a rotating frame of reference. For $2 \le \Omega \le 4$ we keep one real root but also have an imaginary root. This means the system is unstable. If we were looking at a real piece of turbomachinery, we'd have stable operation as we spun up to speed, go through an unstable region and, if we didn't experience failure, re-establish stability at a higher rotational speed.
We can look at the case of problem 4.18 by setting $\omega_1 = \omega_2$. If we let $\omega_1 = \omega_2 = 3$ we'll get



This matches the results of 4.18. For that problem we saw that we had $\ddot{x} + \omega_n^2 x = 0$, $\ddot{y} + \omega_n^2 y = 0$ i.e. both

4.29 (CONT)

THE EIGENVECTOR ASSOCIATED WITH ω_1 SHOULD BE DETERMINED BY A STATIC ANALYSIS: THE MASSLESS I_2 MOVES ACCORDING TO THE ROTATION OF ITS TWO SURROUNDING SPRINGS.

A STATIC BALANCE GIVES

$$k_{\theta_2}(\theta_1-\theta_2) = k_{\theta_3}\theta_2 \Rightarrow \theta_2 = \theta_1 \frac{k_{\theta_2}}{k_{\theta_2}+k_{\theta_3}}$$

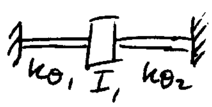
$$\text{EIGENVECTOR}: \begin{Bmatrix} 1 \\ \dfrac{k_{\theta_2}}{k_{\theta_2}+k_{\theta_3}} \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ .706 \end{Bmatrix}$$

NUMERICALLY, MATLAB GIVES, FOR $I_2 = .01$,

$$\omega_1 = 339\ , \quad \underline{X} = \begin{Bmatrix} .8168 \\ .5769 \end{Bmatrix} = 1.224 \begin{Bmatrix} 1.000 \\ .706 \end{Bmatrix},$$

MATCHING OUR PREDICTIONS.

AS $I_2 \to \infty$ WE'LL HAVE A SINGLE DOF SYSTEM



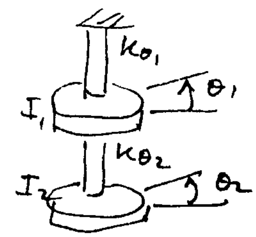
AND A LOW FREQUENCY MODE ASSOCIATED WITH MOVEMENT OF I_2 WITH I_1 PULLED ALONG ACCORDING TO A STATIC MOMENT BALANCE

THE FINITE FREQUENCY RESPONSE HAS A NATURAL FREQUENCY OF $\sqrt{\dfrac{k_{\theta_1}+k_{\theta_2}}{I_1}} = \sqrt{\dfrac{20\times10^5}{10}} = 447$ RAD/S,

WHICH ω_2 IS MOVING TOWARD AS $\omega_2 \to \infty$. THE ω_1 FREQUENCY IS DROPPING TO ZERO, AS EXPECTED

4.34 $I_1\ddot{\theta}_1 = -(k_{\theta_1}+k_{\theta_2})\theta_1 + k_{\theta_2}\theta_2$

$I_2\ddot{\theta}_2 = -k_{\theta_2}\theta_2 + k_{\theta_2}\theta_1$



$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_{\theta_1}+k_{\theta_2} & -k_{\theta_2} \\ -k_{\theta_2} & k_{\theta_2} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(-\omega^2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 10{,}000 & -6000 \\ -6000 & 6000 \end{bmatrix}\right)\begin{Bmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

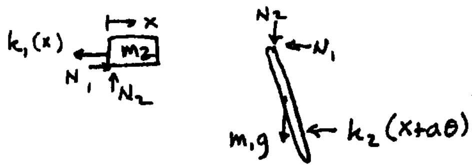
charact. eq: $2\omega^4 - 26\times10^3\omega^2 + 24\times10^6 = 0$

$$\omega^4 - 13\times10^3\omega^2 + 12\times10^6 = 0$$

$$(\omega^2 - 12\times10^3)(\omega^2 - 10^3) = 0$$

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{1000} = 31.6 \text{ rad/s} \\ \omega_2 &= \sqrt{12{,}000} = 109.5 \text{ rad/s} \end{aligned}}$$

4.42



$$m_2\ddot{x} = N_1 - k_1 x \quad (1)$$

$$m_1\left(\ddot{x} + \tfrac{l}{2}\ddot{\theta} - \tfrac{l}{2}\dot{\theta}^2\sin\theta\right) = -k_2(x+a\theta) - N_1 \quad (2)$$

(The first 2 Eq's are $f_x = ma_x$ for m_1 & m_2) Finally we'll sum moments about the center of mass of the bar ($\bar{I} = I$ about center of mass)

$$\bar{I}\ddot{\theta} = -k_2(x+a\theta)\left(a-\tfrac{l}{2}\right) + N_1\tfrac{l}{2} + N_2\tfrac{l}{2}\theta \quad (3)$$

and apply $f_y = ma_y$ to the bar: $-N_2 - m_1 g = \frac{l}{2}\dot{\theta}^2 m_1\cos\theta$ (4)

Using $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$, and combining $\frac{l}{2}$ times Eq 2 to Eq 4 gives

$$m_1\tfrac{l}{2}\ddot{x} + m_1\tfrac{l^2}{4}\ddot{\theta} + \theta\left(N_2\tfrac{l}{2} + m_1 g\, l/2\right) + k_2\tfrac{l}{2}(x+a\theta) + N_1\tfrac{l}{2} = 0 \quad (5)$$

Eq 3 & Eq 5 give $\ddot{\theta}\left(\bar{I} + m_1\frac{l^2}{4}\right) + m_1\frac{l}{2}\ddot{x} + k_2 a x + (k_2 a^2 + m_1 g l)\theta = 0$

Eq 1 & Eq 2 give $(m_1+m_2)\ddot{x} + m_1\frac{l}{2}\ddot{\theta} + (k_1+k_2)x + k_2 a\theta = 0$

$$\begin{bmatrix} m_1+m_2 & m_1\frac{l}{2} \\ m_1\frac{l}{2} & \bar{I}+m_1 l^2/4 \end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\end{Bmatrix} + \begin{bmatrix} k_1+k_2 & k_2 a \\ k_2 a & k_2 a^2 + m_1 g l \end{bmatrix}\begin{Bmatrix} x \\ \theta\end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

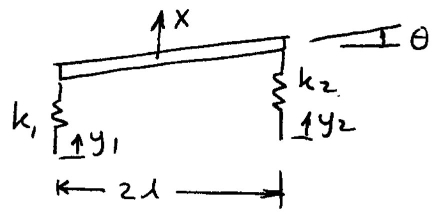
Using the given values: $\left(\bar{I} = \frac{ml^2}{12} = .041\bar{6}\right)$

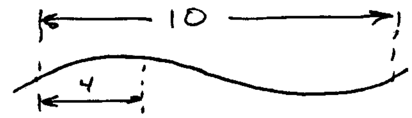
$$\begin{bmatrix} 5 & .5 \\ .5 & .1\bar{6} \end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\end{Bmatrix} + \begin{bmatrix} 300 & 60 \\ 60 & 27.81 \end{bmatrix}\begin{Bmatrix} x \\ \theta\end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\omega_1 = 6.82$ rad/s, $\begin{Bmatrix}\bar{x}\\ \bar{\theta}\end{Bmatrix}_1 = \begin{Bmatrix} .4792 \\ -.8777 \end{Bmatrix}$

$\omega_2 = 13.2$ rad/s, $\begin{Bmatrix}\bar{x}\\ \bar{\theta}\end{Bmatrix}_2 = \begin{Bmatrix} -.0476 \\ .9989 \end{Bmatrix}$

4.47





$$m\ddot{x} + 2kx = k(y_1 + y_2)$$

$$I\ddot{\theta} + 2l^2 k\theta = kl(y_2 - y_1)$$

$$y_1 = 0.02\cos(\omega t)$$

$$y_2 = 0.02\cos(\omega t + .8\pi)$$

$$\frac{x}{2\pi} = \frac{4}{10} \Rightarrow x = .8\pi$$

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 2l^2k \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} .02k\cos(\omega t) + .02k\cos(\omega t + .8\pi) \\ .02kl\cos(\omega t + .8\pi) - .02kl\cos(\omega t) \end{Bmatrix}$$

$$\begin{bmatrix} 1500 & 0 \\ 0 & 2000 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 23{,}394 & 0 \\ 0 & 93{,}576 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 234 \\ -468 \end{Bmatrix}\cos(\omega t) + \begin{Bmatrix} 234 \\ 468 \end{Bmatrix}\cos(\omega t + .8\pi)$$

$$\omega_1 = \sqrt{\frac{23394}{1500}} = 3.95 \text{ RAD/S}$$

$$\omega_2 = \sqrt{\frac{93{,}576}{2000}} = 6.84 \text{ RAD/S}$$

THE LOWER FREQUENCY, ω_1, IS ASSOCIATED WITH $\boxed{\text{VERTICAL TRANSLATIONAL MOTION.}}$ SINCE IT'S THE LOWEST FREQUENCY OF THE TWO, IT WILL BE THE FIRST TO BE EXCITED AS THE CAR'S VELOCITY INCREASES.

$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{3.95} = 1.59$ s . RESONANCE OCCURS WHEN THE CAR TAKES 1.59 S TO TRAVEL 10 m. THE CRITICAL SPEED IS THEREFORE

$$\boxed{V = \frac{10}{1.59} = 6.3 \text{ m/s} = 14 \text{ mph}}$$

4.50 (CONT) TO SEE IF THE MOTION IS EVER PURELY TRANSLATIONAL WE COULD LET $\begin{Bmatrix} x(t) \\ \theta(t) \end{Bmatrix} = \begin{Bmatrix} X \\ \Theta \end{Bmatrix} \cos(\omega t)$ AND ASK IF $\Theta = 0$ EVER SATISFIES OUR GOVERNING EQUATIONS:

$$\left[\begin{bmatrix} 25000 & -10,000 \\ -10,000 & 100,000 \end{bmatrix} - \omega^2 \begin{bmatrix} 1150 & -300 \\ -300 & 1933.\bar{3} \end{bmatrix}\right] \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 100 \\ -1000 \end{Bmatrix}$$

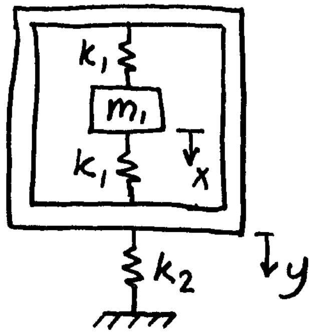
WE COULD SOLVE THESE EQUATIONS FOR DIFFERENT VALUES OF ω AND CHECK IF $\Theta = 0$ FOR A PARTICULAR ω. OR, WE CAN MULTIPLY THE FIRST EQUATION BY 10 AND ADD IT TO THE SECOND EQUATION TO GET

$$240,000\, X - 11,200\omega^2 X = 0 \text{ , WHICH}$$

HAS A SOLUTION $\boxed{\omega = 4.6291 \text{ RAD/S}}$

THUS, IF THE TIME TO TRAVERSE ONE WAVELENGTH OF THE ROAD IS EQUAL TO $\frac{2\pi}{4.6291} = 1.357\text{ S}$, THEN THE RESPONSE WILL BE PURELY TRANSLATIONAL. FOR THIS TO OCCUR $\boxed{V = \frac{8}{1.357} = 5.894 \text{ m/s}}$

4.55



$$m_1: \; m_1\ddot{x} + 2k_1 x - 2k_1 y = 0$$

$$m_2: \; m_2\ddot{y} + (2k_1 + k_2)y - 2k_1 x = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix}\ddot{x} \\ \ddot{y}\end{Bmatrix} + \begin{bmatrix} 2k_1 & -2k_1 \\ -2k_1 & 2k_1 + k_2 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 75 & 0 \\ 0 & 10{,}000 \end{bmatrix}\begin{Bmatrix}\ddot{x} \\ \ddot{y}\end{Bmatrix} + \begin{bmatrix} 147{,}000 & -147{,}000 \\ -147{,}000 & 3{,}447{,}000 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\omega_1 = 18.08$ $\qquad\qquad$ $\omega_2 = 44.47$

$$X_1 = \begin{Bmatrix} .7683 \\ .6401 \end{Bmatrix} \qquad X_2 = \begin{Bmatrix} 1.0000 \\ -.0090 \end{Bmatrix}$$

X_1: BUS MODE. DRIVER CARRIED ALONG WITH LITTLE DYNAMIC CONTRIBUTION - LOW FREQ. MODE

X_2: DRIVER MODE. BUS ESSENTIALLY UNAFFECTED. HIGH FREQ MODE

IF BUS DROPPED FROM 2 METERS WE HAVE

$$g\,dx = \frac{v^2}{2} \Rightarrow V = 6.26 \text{ m/s}$$

STATIC EQUILIBRIA: BUS: $k_2 y_{eq} = (m_1 + m_2)g \Rightarrow y_{eq} = .03\text{ m}$

DRIVER: $2k_1 \Delta x_{eq} = m_1 g \Rightarrow x_{eq} = .005\text{ m}$. SINCE DRIVER'S ABSOLUTE EQUIL. POSITION WILL VARY WITH BUS POSITION, WE HAVE $x_{eq}\big|_{abs} = .03 + .005 = .035\text{ m}$

THUS $\begin{Bmatrix} x(0) \\ y(0) \end{Bmatrix} = \begin{Bmatrix} -.035 \\ -.03 \end{Bmatrix}$, $\begin{Bmatrix} \dot{x}(0) \\ \dot{y}(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} 6.26$

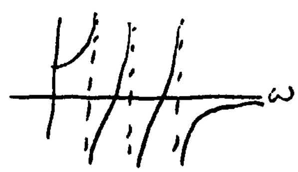
GENERAL SOLUTION:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = a_1 X_1 \sin(18.08t) + a_2 X_1 \cos(18.08t) + a_3 X_2 \sin(44.47t) + a_4 X_2 \cos(44.47t)$$

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = 18.08(a_1 X_1 \cos(18.08t) - a_2 X_1 \sin(18.08t)) + 44.47(a_3 X_1 \cos(44.47t) - a_4 X_2 \sin(44.47t))$$

4.69 If m_2 and m_4 are stationary then m_3 must be stationary. Otherwise, m_3's motion would cause m_4 to move. For m_3 to be stationary, m_1 must be also, by the same reasoning. Thus all the masses are stationary. Therefore there are no forcing frequencies that allow x_2 & x_4 to equal zero since there are no oscillations for f to counter.

4.70 If the forcing and response are associated with the same mass we have collocated sensing/actuation. The transfer function will look like



3 DOF system, for example

i.e. n-1 zeros for an n-DOF system. The smallest number of zeros is zero, associated with a response like

which corresponds to maximum non-collocation.

4.72
(CONT) TO see what's happening, let's look at the unforced system:

$$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 1200 & 100 \\ 100 & 300 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This has natural frequencies and eigenvectors:

$$\omega_1^2 = 250 \;, \quad \begin{Bmatrix} \bar{X} \\ \bar{\theta} \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} \quad \text{and} \quad \omega_2^2 = 350 \;, \quad \begin{Bmatrix} \bar{X} \\ \bar{\theta} \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

Thus our model matrix $U = \begin{bmatrix} .3536 & .3536 \\ -.7071 & .7071 \end{bmatrix}$ (having mass normalized $\begin{Bmatrix} \bar{X} \\ \bar{\theta} \end{Bmatrix}_1$ and $\begin{Bmatrix} \bar{X} \\ \bar{\theta} \end{Bmatrix}_2$)

$$U^T M U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U^T K U = \begin{pmatrix} 250 & 0 \\ 0 & 350 \end{pmatrix}$$

and $U^T \begin{Bmatrix} 1 \\ -.5 \end{Bmatrix} = \begin{Bmatrix} .7071 \\ 0 \end{Bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{Bmatrix} + \begin{bmatrix} 250 & 0 \\ 0 & 350 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{Bmatrix} .7071 \\ 0 \end{Bmatrix}$$

Thus we can see that the forcing <u>isn't</u> <u>exciting</u> the second mode

4.78
$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 500 & -50 & 0 \\ -50 & 400 & -20 \\ 0 & -20 & 100 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$[v,d] = eig(K, M)$ YIELDS (MATLAB VERSION 6.1)

$$V = \begin{bmatrix} .0058 & -.3153 & -.0233 \\ .0293 & -.0516 & .7046 \\ .4995 & .0107 & -.0200 \end{bmatrix} \quad d = \begin{bmatrix} 24.7065 & 0 & 0 \\ 0 & 49.1824 & 0 \\ 0 & 0 & 201.1111 \end{bmatrix}$$

$$[V]^T[M][V] = I, \quad [V]^T[K][V] = \begin{bmatrix} 24.7065 & 0 & 0 \\ 0 & 49.1824 & 0 \\ 0 & 0 & 201.1111 \end{bmatrix}$$

BOTH MATRICES ARE DIAGONAL $\Rightarrow$ EIGENVECTORS SATISFY ORTHOGONALITY

4.79 WE KNOW $\vec{X}_3^T[M]\vec{X}_1 = 0$ AND $\vec{X}_3^T[M]\vec{X}_2 = 0$

LET $\vec{X}_3 = \begin{Bmatrix} 1 \\ b \\ c \end{Bmatrix}$

$$(1 \quad b \quad c)\begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} = 0 \Rightarrow 1 + c + \sqrt{2}\, b = 0 \qquad (1)$$

$$(1 \quad b \quad c)\begin{Bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{Bmatrix} = 0 \Rightarrow \sqrt{2} - \sqrt{2}\, c = 0 \qquad (2)$$

$(2) \Rightarrow \quad c = 1 \qquad (3)$

$(3) \to (1) \Rightarrow \quad 2 + \sqrt{2}\, b = 0 \Rightarrow b = -\sqrt{2}$

$$\boxed{\vec{X}_3 = \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}}$$

4.87 $\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\end{Bmatrix} + \begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix} = \begin{Bmatrix}2\\3\end{Bmatrix}\cos 2t$

Free vib: $\begin{bmatrix}1&0\\0&1\end{bmatrix}\ddot{X} + \begin{bmatrix}2&-1\\-1&2\end{bmatrix}X = \begin{Bmatrix}0\\0\end{Bmatrix}$ Solving gives us

$\omega_1 = 1$, $X_1 = \begin{Bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{Bmatrix}$ (mass normalized)

$\omega_2 = \sqrt{3}$, $X_2 = \begin{Bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{Bmatrix}$

$U = \begin{bmatrix}1&1\\1&-1\end{bmatrix}1/\sqrt{2}$

$U^T\begin{bmatrix}1&0\\0&1\end{bmatrix}U = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $U^T\begin{bmatrix}2&-1\\-1&2\end{bmatrix}U = \begin{bmatrix}1&0\\0&3\end{bmatrix}$, $U^T\begin{Bmatrix}2\\3\end{Bmatrix} = \frac{1}{\sqrt{2}}\begin{Bmatrix}5\\-1\end{Bmatrix}$

$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{Bmatrix}\ddot{\eta}_1\\\ddot{\eta}_2\end{Bmatrix} + \begin{bmatrix}1&0\\0&3\end{bmatrix}\begin{Bmatrix}\eta_1\\\eta_2\end{Bmatrix} = \frac{1}{\sqrt{2}}\begin{Bmatrix}5\\-1\end{Bmatrix}c\,2t$

1st mode: $\ddot{\eta}_1 + \eta_1 = \frac{5}{\sqrt{2}}\cos(2t)$

If $\eta_1 = \bar{\eta}_1 \cos 2t$ then

$(-4+1)\bar{\eta} = \frac{5}{\sqrt{2}}$, $-3\bar{\eta} = \frac{5}{\sqrt{2}}$, $\bar{\eta} = \frac{-5}{3\sqrt{2}}$

Thus the response is $-\frac{5}{3\sqrt{2}}\cos(2t)$ and the magnitude of the modal response is $\frac{5}{3\sqrt{2}}$

4.91 (cont)

THE FIRST COLUMN OF [VV] IS

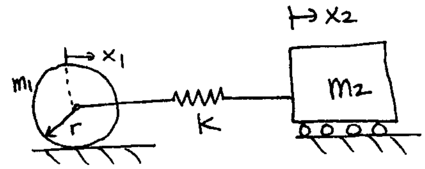
$$\begin{Bmatrix} .0529+.3343i \\ -.0274-.1735i \\ -.0390-.2463i \\ .0256+.1622i \\ -1 \\ .5191 \\ .7369 \\ -.4851 \end{Bmatrix}$$

$$\mathrm{ANGLE}(V1) = \begin{Bmatrix} 1.414 \\ -1.7276 \\ -1.7276 \\ 1.414 \\ 3.1416 \\ 0 \\ 0 \\ -3.1416 \end{Bmatrix}$$

$$V1 * \exp(-j * 1.414) = \begin{Bmatrix} .3385 \\ -.1757 \\ -.2494 \\ .1642 \\ -.1562+.9877i \\ .0811-.5127i \\ .1151-.7278i \\ -.0758+.4791i \end{Bmatrix}$$

WHICH HAS REAL DISPLACEMENT ENTRIES, AS EXPECTED

4.97



Rotation rate of the cylinder, $\dot{\theta}$, is equal to $\frac{\dot{x}_1}{r}$ from the no-slip condition.

$$KE_{cylinder} = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m_1 \dot{x}_1^2 = \frac{1}{2}\left(\frac{mr^2}{2}\right)\frac{\dot{x}_1^2}{r^2} + \frac{1}{2} m_1 \dot{x}_1^2$$

$$KE_{m_2} = \frac{1}{2} m_2 \dot{x}_2^2$$

$$KE = \frac{1}{2}\left[m_1 + \frac{m_1}{2}\right]\dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}^2$$

$$PE = \frac{1}{2} k (x_2 - x_1)^2$$

$$L = KE - PE$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0 \Rightarrow \left(m_1 + \frac{m_1}{2}\right)\ddot{x}_1 + k(x_1 - x_2) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0 \Rightarrow m_2 \ddot{x}_2 + k(x_2 - x_1) = 0$$

$$\begin{bmatrix} \frac{3m_1}{2} & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1500 & 0 \\ 0 & 1500 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\omega_1 = 0, \quad X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \omega_2 = 0.8165, \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

4.101 (cont)

$$\lambda_2^2 = 8 \qquad \begin{bmatrix} -4 & -4 & 0 & 0 \\ -4 & -4 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & -4 & -4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

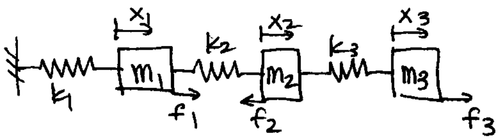
LET $\underline{\xi}_2 = \begin{Bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{Bmatrix}$ AND USE $\underline{\xi}_4 = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix}$ FOR λ_4^2.

OUR MODAL MATRIX $[U]$ IS

$$[U] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

THE EIGENVECTORS $\underline{\xi}_i$ ARE INDEPENDENT AND SPAN THE SYSTEM'S CONFIGURATION SPACE

4.107



$$\begin{pmatrix} f_1^{x_1} \\ f_2^{x_1} \\ f_3^{x_1} \end{pmatrix} = \begin{pmatrix} k_1+k_2 \\ k_2 \\ 0 \end{pmatrix} x_1 \; ; \; \begin{pmatrix} f_1^{x_2} \\ f_2^{x_2} \\ f_3^{x_2} \end{pmatrix} = \begin{pmatrix} -k_2 \\ -k_2-k_3 \\ -k_3 \end{pmatrix} x_2 \; ; \; \begin{pmatrix} f_1^{x_3} \\ f_2^{x_3} \\ f_3^{x_3} \end{pmatrix} = \begin{pmatrix} 0 \\ k_3 \\ k_3 \end{pmatrix} x_3$$

$$\underbrace{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}}_{F} = \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ k_2 & -k_2-k_3 & k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}}_{[K]} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{X}$$

$$X = [K]^{-1} F$$

Using the given values we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 300 & -200 & 0 \\ 200 & -250 & 50 \\ 0 & -50 & 50 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.09 \\ 0.085 \\ 0.105 \end{bmatrix}$$

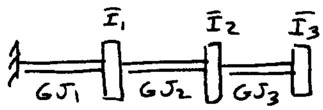
Checking at left & right of each mass:

m_1: $k_2(x_2-x_1)-k_1x_1 = 200(0.085-0.09)-100(0.09) = -10N$ ✓

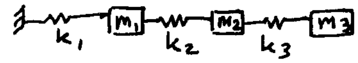
m_2: $k_3(x_3-x_2)-k_2(x_2-x_1) = (0.105-0.085)50-200(0.085-0.09) = 2N$ ✓

m_3: $-k_3(x_3-x_2) = -50(0.105-0.085) = -1N$ ✓

4.110



FROM APP B, $k_{eq} = \frac{GJ}{l}$. THUS WE CAN VIEW THE SYSTEM AS

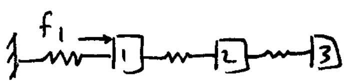


WHERE $k_i = \frac{GJ_i}{l_i}$, $m_i = \bar{I}_i$ AND $\theta_i \to x_i$, $M_i \to f_i$

$$\begin{Bmatrix} f_1^{x_1} \\ f_2^{x_1} \\ f_3^{x_1} \end{Bmatrix} = \begin{Bmatrix} k_1+k_2 \\ -k_2 \\ 0 \end{Bmatrix} x_1, \quad \begin{Bmatrix} f_1^{x_2} \\ f_2^{x_2} \\ f_3^{x_2} \end{Bmatrix} = \begin{Bmatrix} -k_2 \\ k_2+k_3 \\ -k_3 \end{Bmatrix}, \quad \begin{Bmatrix} f_1^{x_3} \\ f_2^{x_3} \\ f_3^{x_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -k_3 \\ k_3 \end{Bmatrix} x_3$$

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \Rightarrow [k] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

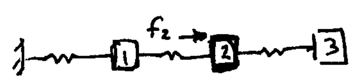
NOW APPLY $f_1 = f_1$, $f_2 = f_3 = 0$



$$x_1 = \frac{f_1}{k_1}, \quad x_2 = x_3 = x_1 = \frac{f_1}{k_1}$$

$$\begin{Bmatrix} x_1^{f_1} \\ x_2^{f_1} \\ x_3^{f_1} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{k_1} \\ \frac{1}{k_1} \\ \frac{1}{k_1} \end{Bmatrix} f_1$$

APPLY $f_1 = f_3 = 0$, $f_2 = f_2$



EQUIV SPRING AT m_2: $\frac{k_1 k_2}{k_1+k_2} \Rightarrow x_2 = \frac{f_2}{\frac{k_1 k_2}{k_1+k_2}} = \left(\frac{k_1+k_2}{k_1 k_2}\right) f_2$

$$x_3 = x_2 = \left(\frac{k_1+k_2}{k_1 k_2}\right) f_2$$

A FORCE BALANCE AT m_1 YIELDS $k_1 x_1 = (x_2 - x_1) k_2$ WHICH IMPLIES (USING $x_2 = \frac{k_1+k_2}{k_1 k_2} f_2$) THAT

$$x_1 = \frac{f_2}{k_2}$$

$$\begin{Bmatrix} x_1^{f_2} \\ x_2^{f_2} \\ x_3^{f_2} \end{Bmatrix} = \begin{Bmatrix} 1/k_2 \\ (k_1+k_2)/(k_1 k_2) \\ (k_1+k_2)/(k_1 k_2) \end{Bmatrix} f_2$$