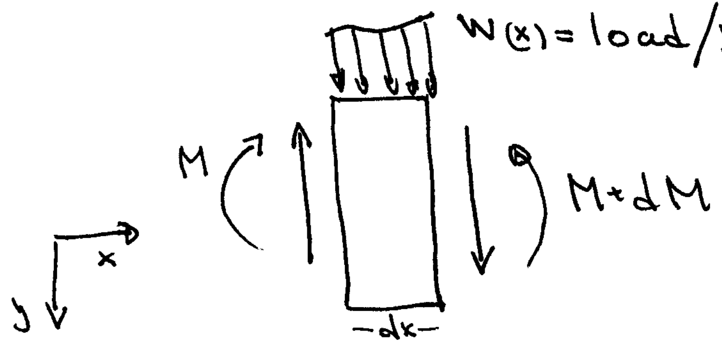
Lateral Vibrations of a Beam

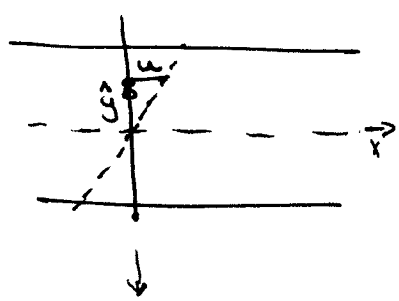


$$\Sigma F_y = m \cdot \frac{\partial^2 y}{\partial t^2} \Rightarrow W \cdot dx + dV = \rho A\, dx \Rightarrow W = -\frac{\partial V}{\partial x} + \rho A \frac{\partial^2 y}{\partial t^2} \quad (1)$$

$$\Sigma M_{cm} = 0 \Rightarrow V \cdot \frac{dx}{2} + (V + dV)\frac{dx}{2} = -dM \Rightarrow V(x,t) = -\frac{\partial M}{\partial x} \quad (2)$$

$$(2) \rightarrow (1) \Rightarrow W(x,t) = \frac{\partial^2 M}{\partial x^2} + \rho A \frac{\partial^2 y}{\partial t^2} \quad (3)$$

Beam theory: plane sections remain plane



$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x} \cdot \hat{y}\right) = \frac{\partial^2 y}{\partial x^2} \cdot \hat{y}$$

$$\text{Moment} = M = \int_A \sigma_x \cdot \hat{y} \cdot dA =$$

$$M = \int_A E \frac{\partial^2 y}{\partial x^2} \cdot \hat{y}^2\, dA = E \frac{\partial^2 y}{\partial x^2} \int_A \hat{y}^2\, dA = EI \frac{\partial^2 y}{\partial x^2} \quad (4)$$

$$(4) \rightarrow (3) \Rightarrow W(x,t) = \frac{\partial^2}{\partial x^2}\left[EI \frac{\partial^2 y}{\partial x^2}\right] + \rho A \frac{\partial^2 y}{\partial t^2}$$

Equ of motion:

$$\boxed{\rho A \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI \frac{\partial^2 y}{\partial x^2}\right] = W}$$

If EI is constant and $W=0$

$$\boxed{EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = 0}$$

Solution: Use separation of variables

Let $y(x,t) = X(x)\,T(t)$ $\Rightarrow$

$$\frac{d^4 X}{dx^4}\cdot T(t) + \frac{\rho A}{EI}\frac{d^2 T}{dt^2}X(x) = 0 \Rightarrow$$

$$\underbrace{-\frac{T(t)}{\frac{d^2T}{dt^2}}}_{f(t)} = \underbrace{\frac{\rho A}{EI}\frac{X(x)}{\frac{d^4 X(x)}{dx^4}}}_{g(x)}$$

$f(t) = g(x) \Rightarrow f(t) = g(x) = \text{constant}$

call this constant $\frac{1}{\omega^2}$

$$\Rightarrow \begin{cases} (1)\; \frac{d^2T}{dt} + \omega^2 T(t) = 0 & \Rightarrow T(t) = E\sin\omega t + F\cos\omega t \\ (2)\; \frac{d^4 X}{dx^4} - \beta^4 X(x) = 0 & \text{where } \beta^4 = \frac{\rho A\omega^2}{EI} \end{cases}$$

Let $X(x) = e^{\lambda x}$ subst into (2) $\Rightarrow \lambda = \pm\beta, \pm i\beta$

so $X(x) = c_1 e^{\beta x} + c_2 e^{-\beta x} + c_3 e^{i\beta x} + c_4 e^{-i\beta x}$

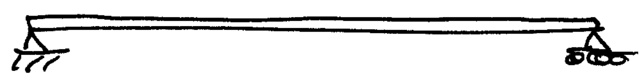
or $X(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$

where $\sinh\beta x = \frac{e^{\beta x} - e^{-\beta x}}{2}$, $\cosh = \frac{e^{\beta x} + e^{-\beta x}}{2}$

We have 6 unknowns

But we also have boundary conditions
and 2 initial conditions.

<u>Ex</u>



Pinned - Pinned beam

$y(0,t) = 0$

$y(l,t) = 0$

$M(0,t) = \left.\dfrac{\partial^2 y}{\partial x^2}\right|_{x=0} = 0$

$M(l,t) = \left.\dfrac{\partial^2 y}{\partial x^2}\right|_{x=l} = 0$

where

$$\frac{\partial^2 y}{\partial x^2} = T(t)\left[-A\beta^2 \sin\beta x - B\beta^2 \cos\beta x + C\beta^2 \sinh\beta x + D\beta^2 \cosh\beta x\right]$$

at $x=0$ $\quad y(0,t)=0 \Rightarrow B+D=0$

$\left.\dfrac{\partial^2 y}{\partial x^2}\right|_{x=0} \Rightarrow -B+D=0 \quad \Rightarrow \boxed{B=D=0}$

at $x=l$ $\quad y(l,t)=0=A\sin\beta l + C\sinh\beta l$

$\left.\dfrac{\partial^2 y}{\partial x^2}\right|_{x=l} = 0 = A\sin\beta l + C\sinh\beta l$

$$\Rightarrow \begin{cases} A\sin\beta l = 0 \\ C\sin\beta l = 0 \end{cases}$$

But $\sin\beta l > 0 \quad \forall \beta l > 0$

$\Rightarrow \boxed{C = 0}$

and $A \sin\beta l = 0 \quad \Rightarrow$

$A = 0 \Rightarrow$ trivial solution

or

$\sin\beta l = 0 \Rightarrow \beta_n \cdot l = n \cdot \pi \Rightarrow \beta_n = \frac{n\pi}{l}$

$\Rightarrow \boxed{\omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}}}$ $\left(\text{since } \beta^4 = \frac{\rho A \omega^2}{EI}\right)$

natural frequency of the beam

and since $X(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$

we get $X_n(x) = A_n \sin\beta_n x$ ← n^{th} mode shape

or

$\boxed{X_n(x) = A_n \sin\frac{n\pi}{l} x}$

and the displacement of the beam:

$$y = \sum_{s=1}^{\infty} X_s T_s = \sum_{s=1}^{\infty} \sin\frac{n\pi}{l} x \left(E \sin\omega_s t + F\cos\omega_s t\right)$$

E and F are determined the same way as for the bar. I.e. given the initial shape and velocity, $y(x,0) = H(x)$ and $\dot{y}(x,0) = G(x)$ multiply each side by $\sin\beta_r x$ and integrate over the length.

Forced Oscillation of a Beam

Equ of motion

$$EI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = W(x,t)$$

Let $y(x,t) = \sum_{n=1}^{\infty} T_n(t) \cdot X_n(t)$

modes

Substitute into DEQ:

$$\sum_{n=1}^{\infty} \left[T_n(t)\, EI \frac{d^4 X_n}{dx^4} + \rho A\, \ddot{T}_n(t) \cdot X_n \right] = W(x,t) \qquad (1)$$

But from the unforced problem we know:

$$\frac{d^4 X_n}{dx^4} - \beta_n^4 X_n = 0 \Rightarrow \frac{d^4 X_n}{dx^4} = \beta_n^4 X_n$$

$$\Rightarrow \quad \frac{d^4 X_n}{dx^4} = \frac{\rho A}{EI} \omega_n^2 X_n$$

so equation (1) becomes

$$\sum_{n=1}^{\infty} \rho A X_n \left(\ddot{T}_n + \omega_n^2 T_n \right) = W(x,t)$$

Multiply by X_m and integrate over the length

$$\sum_{n=1}^{\infty} \left(\ddot{T}_n + \omega_n^2 T_n \right) \cdot \underbrace{\int_0^l \rho A X_n(x) \cdot X_m(x)\, dx}_{\substack{=0 \quad n \neq m \\ \neq 0 \quad n = m}} = \int_0^l X_m(x) \cdot W(x,t) \cdot dx$$

$$\Rightarrow \quad \ddot{T}_m + \omega_m^2 T = \frac{\int_0^l \Phi_m(x) \cdot W(x,t)\,dx}{\int_0^l \rho A \Phi_m^2(x)\,dx} = f_m(t) \quad , m=1,2\ldots$$

$\Rightarrow$

$$T_m = A_m \sin \omega_m \cdot t + B_m \cdot \cos \omega_m t + \frac{1}{\rho A \omega_m} \int_0^t f(\tau) \cdot \sin \omega_m (t-\tau)\,d\tau$$

The A's and B's are determined by the initial conditions.

So the motion of the beam becomes:

$$y(x,t) = \sum_{n=1}^{\infty} \left\{ \left[A_n \sin \omega_n t + B_n \cos \omega_n t + \frac{1}{\rho A \omega_n} \int_0^t f(\tau) \cdot \sin \omega_n (t-\tau)\,d\tau \right] \Phi_n \right\}$$