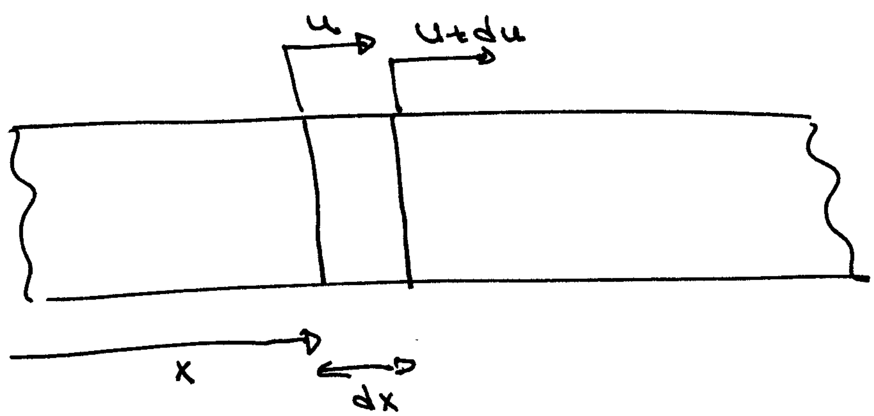
Vibration of Elastic Bodies

Longitudinal Vibration of Uniform Bars or Rods

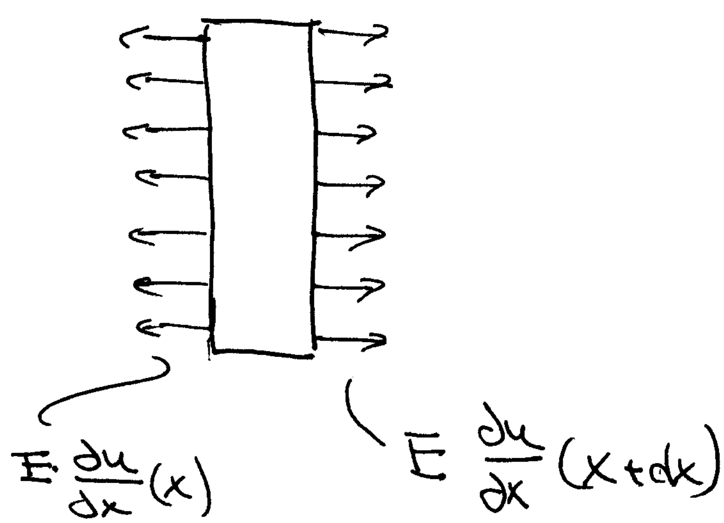


Area = A = const
Modulus of elasticity = E
Density = ρ

Strain : $\epsilon_x = \frac{\partial u}{\partial x}$

Stress : $\sigma_x = E \cdot \epsilon_x = E \frac{\partial u}{\partial x}$

Assume plane sections remain plane



$$\Sigma F_x = m \cdot a \quad \Longrightarrow$$

$$AE \cdot \frac{\partial u}{\partial x}(x+dx) - AE\frac{\partial u}{\partial x}(x) = \overbrace{\rho A\, dx}^{m} \cdot \frac{\partial^2 u}{\partial t^2} \qquad (1)$$

Expand $\frac{\partial u}{\partial x}(x+dx)$ about x:

$$\frac{\partial u}{\partial x}(x+dx) = \frac{\partial u}{\partial x}(x) + dx\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \ldots$$

$$= \frac{\partial u}{\partial x}(x) + \frac{\partial^2 u}{\partial x^2}(x) \cdot dx + \ldots$$

so (1) $\rightarrow$

$$AE\frac{\partial u}{\partial x} + AE\frac{\partial^2 u}{\partial x^2}dx + -AE\frac{\partial u}{\partial x}(x) = \rho A\frac{\partial^2 u}{\partial t^2} \cdot dx$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}} \quad \leftarrow \text{Eq of motion}$$

1 dimensional wave equation

To generate solutions let

$$u(x,t) = X(x)\,T(t)$$

substitute into the wave equation

$$\frac{d^2 T(t)}{dt^2} X(x) = \frac{E}{\rho} T(t) \cdot \frac{d^2 X}{dx^2}$$

$$\Rightarrow \frac{\frac{d^2 T}{dt^2}}{T(t)} = \frac{E}{\rho} \frac{\frac{d^2 X}{dx^2}}{X}$$

so

$$(*) \quad \frac{d^2 T}{dt^2} \Big/ T(t) = -\omega^2 = \text{const}$$

$$(**) \quad \frac{\frac{d^2 X}{dx^2}}{X} = \frac{E}{\rho} = -\omega^2$$

$$(*) \rightarrow \frac{d^2 T}{dt^2} + \omega^2 T(t) = 0 \Rightarrow$$

$$\Rightarrow \boxed{T(t) = A \sin \omega t + B \cos \omega t}$$

$$(**) \rightarrow \frac{d^2 X}{dx^2} + \frac{\rho}{E} \omega^2 X = 0$$

$$\Rightarrow \boxed{X(x) = C \sin \omega \sqrt{\frac{\rho}{E}}\, x + D \cos \omega \sqrt{\frac{\rho}{E}} \cdot x}$$

Boundary conditions are needed to determine C and D

(IC's are needed to determine A + B)

Free end: $\sigma_{end} = 0 \Rightarrow \epsilon_{end} = 0 \Rightarrow \left.\frac{\partial u}{\partial x}\right|_{end} = 0$

Fixed end: $u_{end} = 0$

End with mass attached: $F = M \cdot a \Rightarrow$

$$M \left.\frac{\partial^2 u}{\partial t^2}\right|_{end} + EA \left.\frac{\partial u}{\partial x}\right|_{end} = 0$$

End with spring attached:

$$K \cdot u\big|_{end} + EA \left.\frac{\partial u}{\partial x}\right|_{end} = 0$$