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Solution of $M\ddot{\underset{\sim}{x}} + K\underset{\sim}{x} = 0$

Normal Mode Approach

Let $\underset{\sim}{x} = \underline{X} \cdot e^{i\omega t}$

substitute into DEQ:

$$(-\omega^2 M\underline{X} + K\underline{X})e^{i\omega t} = \underset{\sim}{0}$$

$$(M^{-1}K - \omega^2 I)\underset{\sim}{\underline{X}} = 0 \qquad \text{(eigen value problem)}$$

For nontrivial solutions:

$$\det(M^{-1}K - \omega^2 I) = 0 \leftarrow \text{characteristic equation}$$

The characteristic equation will be an n^{th} order polynomial in ω^2 where n is size the system.

$(M^{-1}K = H$: The Dynamical Matrix$)$

Assume that all of the roots of $\det(M^{-1}K - \omega^2 I)$ are distinct.

Then for each root (eigenvalue), ω_r, there will be a corresponding modal vector (eigenvector) $\underset{\sim}{\underline{X}}^r$

$$(M^{-1}K - \omega_r^2 I)\underline{X}^r = 0$$

solve for $\underline{X}^r$ to get modeshape.

Normal Mode Approach (cont)

The modes are orthogonal w.r.t. M and K

Proof:

For the r^{th} mode we have

$$(M^{-1}K - \omega_r^2 I)\underset{\sim}{X}^r = 0$$

$$K\underset{\sim}{X}^r - \omega_r^2 M\underset{\sim}{X}^r = \underset{\sim}{0}$$

Take the transpose of this equation

$$\underset{\sim}{X}^{rT} K - \omega_r^2 \underset{\sim}{X}^{rT} \cdot M = 0$$

Post multiply by $\underset{\sim}{X}^s$ (the s^{th} mode shape)

$$\underset{\sim}{X}^{rT} K \underset{\sim}{X}^s - \omega_r^2 \underset{\sim}{X}^{rT} M \cdot \underset{\sim}{X}^s = 0 \qquad *$$

Pre multiply the equation

$$K\underset{\sim}{X}^s - \omega_s^2 M\underset{\sim}{X}^s = \underset{\sim}{0}$$

by $\underset{\sim}{X}^{rT}$:

$$\underset{\sim}{X}^{rT} K \underset{\sim}{X}^s - \omega_s^2 \underset{\sim}{X}^{rT} M \underset{\sim}{X}^s = 0 \qquad **$$

Subtract $*$ from $**$:

$$(\omega_r^2 - \omega_s^2)\underset{\sim}{X}^{rT} M \underset{\sim}{X}^s = \underset{\sim}{0}$$

Hence we conclude that for distinct eigenvalues

$$\boxed{\underset{\sim}{X}^{rT} M \underset{\sim}{X}^s = 0 \qquad r \neq s}$$

$\Rightarrow$ $\underset{\sim}{X}^r$ and $\underset{\sim}{X}^s$ are orthogonal w.r.t M

Also $\boxed{\underline{X}^{rT} K \underline{X}^{s} = 0}$ $\quad r \neq s$

$\Rightarrow$ $\underline{X}^r$ and $\underline{X}^s$ are also orthogonal w.r.t K.

Define the modal column matrix

$$A \equiv \left[\underline{X}^1, \underline{X}^2, \cdots \underline{X}^n \right], \qquad n \times n \text{ matrix}$$

It can be seen from $-\omega^2 M \underline{X} + K \underline{X} = 0$ that

$$-MA \begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots & \\ & & & \omega_n^2 \end{bmatrix} + KA = 0$$

or

$$A^{-1} M^{-1} K A = \begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots & \\ & & & \omega_n^2 \end{bmatrix}$$

Hence, the matrix $\underset{\sim}{A}$ represents the coordinate transformation that diagonalizes the dynamical matrix H.

<u>Also</u> since

$$X^{rT} M X^s = 0 \quad r \neq s \quad \text{we get}$$

$$A^T M A = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \qquad X^{rT} M X^r \text{, nonzero diagonal}$$

and

$$A^T K A = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \qquad X^{rT} K X^r$$

In previous example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Test orthogonality:

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & -4 \\ 1 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

The total solution will be

$$\underset{\sim}{x}(t) = \sum_{i=1}^{n} \underset{\sim}{X}^{i} A_i \cos(\omega_i t + \phi_i)$$

Not A; the modal column matrix.

The constants A_i and ϕ_i can be evaluated from the initial conditions

$$\underset{\sim}{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} \quad \text{and} \quad \dot{\underset{\sim}{x}} = \begin{pmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \vdots \\ \dot{x}_n(0) \end{pmatrix}$$

Principle Coordinate Approach

Let us seek a set of coordinates $\underset{\sim}{P}$ in which the system of DEQ's will be uncoupled. Consider the set of coordinates defined by

$$\underset{\sim}{X} = T \underset{\sim}{P}$$

where T is a coordinate transfer matrix.

The DEQ becomes

$$M \ddot{\underset{\sim}{x}} + K \underset{\sim}{x} = 0$$

$$M T \ddot{\underset{\sim}{p}} + K T \underset{\sim}{p} = \underset{\sim}{0}$$

$$\ddot{\underset{\sim}{P}} + T^{-1} M^{-1} K T \underset{\sim}{P} = 0$$

We showed previously that

$$A^{-1}(M^{-1}K)A = \begin{bmatrix} \diagdown & 0 \\ 0 & \diagdown \end{bmatrix} \quad \text{(diagonal matrix)}$$

Hence the system will be uncoupled if we select T such that

$$T = A \cdot \underbrace{\begin{bmatrix} \diagdown & 0 \\ 0 & \diagdown \end{bmatrix}}_{\text{constants}}$$

Then T satisfies the orthogonality condition

$$\boxed{T^T M T = \begin{bmatrix} \diagdown a_i \diagdown \end{bmatrix}}, \quad \boxed{T^T K T^T = \begin{bmatrix} \diagdown b_i \diagdown \end{bmatrix}}$$

so

$$T^T M T \ddot{\underset{\sim}{p}} + T^T K T \underset{\sim}{p} = 0$$

$$a_i \ddot{p}_i + b_i p_i = 0 \qquad i = 1 \ldots n$$

Uncoupled eqns. of motion

Those coordinates p_i are called principle coordinates.