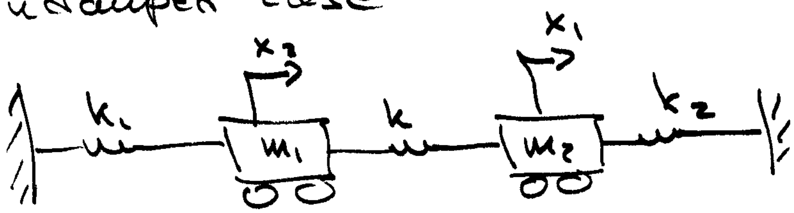
Multi-Degree of Freedom Systems

Two degree of freedom

Free, undamped case



$$T = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 , \quad V = \tfrac{1}{2} k_1 x_1^2 + \tfrac{1}{2} k_2 \cdot x_2^2 + \tfrac{1}{2} k (x_2 - x_1)^2$$

$$L = T - V$$

Lagrang eq: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_i}\right) - \dfrac{\partial L}{\partial x_i} = 0 \qquad i = 1, 2$

$$\Rightarrow \quad \begin{aligned} m_1 \ddot{x}_1 + k_1 x_1 + k x_1 - h x_2 &= 0 \\ m_2 \ddot{x}_2 + h_2 x_2 + k x_2 - k x_1 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} m_1 \ddot{x}_1 + (k_1 + k) x_1 - k x_2 &= 0 \\ m_2 \ddot{x}_2 - k x_1 + (k + k_2) x_2 &= 0 \end{aligned}$$

we can write this as: $\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k_1 + k & -k \\ -k & k + k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\quad M \ddot{\underset{\sim}{x}} + K \underset{\sim}{x} = 0 \quad$ ✱ mass matrix $M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$

stiffness matrix $K = \begin{pmatrix} k_1 + k & -k \\ -k & k + k_2 \end{pmatrix}$

Solution: assume $x_1 = X_1 e^{i\omega t}$
$x_2 = X_2 e^{i\omega t}$ $\Rightarrow \underset{\sim}{x} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t}$

or $\quad \underset{\sim}{x} = \underline{X} e^{i\omega t}$

Plug this into ✱ : $-M\omega^2 \underline{X} + k\underline{X} = 0 \Rightarrow (-M\omega^2 + k)\underline{X} = 0$ ② (✱✱)

or written out: $(k_1 + k - m_1\omega^2)\underline{X}_1 - k\underline{X}_2 = 0$

$$-k\underline{X}_1 + (k_2 + k - m_2\omega^2)\underline{X}_2 = 0$$

A nontrivial solution iff $\det(-M\omega^2 + k) = 0$

Here: $$\begin{vmatrix} k_1 + k - m_1\omega^2 & -k \\ -k & k_2 + k - m_2\omega^2 \end{vmatrix} = 0$$

$\Rightarrow$ equation in ω – characteristic equation or freq. equ.

Expand: $$\omega^4 - \left[\frac{k_1+k}{m_1} + \frac{k_2+k}{m_2}\right]\omega^2 + \frac{k_1k_2 + k_1k + k_2k}{m_1 \cdot m_2} = 0$$

$$\omega^2 = \frac{1}{2}\left[\frac{k_1+k}{m_1} + \frac{k_2+k}{m_2} \pm \sqrt{\left(\frac{k_1+k}{m_1} + \frac{k_2+k}{m_2}\right)^2 - 4\,\frac{k_1k_2 + k_1k + k_2 k}{m_1 m_2}}\right.$$

or

$$\omega^2 = \frac{1}{2}\left[\frac{k_1+k}{m_1} + \frac{k_2+k}{m_2} \pm \sqrt{\underbrace{\left(\frac{k_1+k}{m_1} - \frac{k_2+k}{m_2}\right)^2 + 4\,\frac{k^2}{m_1 m_2}}_{\geq 0}}\right.$$

$\Rightarrow$ ω^2 is real > 0

$\Rightarrow$ we obtain two roots for $\omega^2 \to \omega_1^2$ and ω_2^2

$\Rightarrow$ $\omega = \pm\omega_1, \pm\omega_2$ Let $\omega_1 < \omega_2$

then ω_1 = Fundamental or 1st mode freq., ω_2 = 2nd mode freq mode shapes:

For ω_1: $(-M\omega_1^2 + K)\underset{\sim}{X}^1 = 0 \quad \Rightarrow$

$$\begin{pmatrix} k_1+k-m_1\omega_1^2 & -k \\ -k & k_2+k-m_2\omega_1^2 \end{pmatrix}\begin{pmatrix} X_1^1 \\ X_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let $X_1^1 = 1 \Rightarrow X_2^1 = \dfrac{k_1+k-m_1\omega_1^2}{k} = \mu_1$

so $X^1 = \begin{pmatrix} 1 \\ \mu_1 \end{pmatrix} \leftarrow$ First mode shape

For ω_2: $(-M\omega_2^2 + K)\underset{\sim}{X}^2 = 0 \quad \Rightarrow$

$$\begin{pmatrix} k_1+k-m_1\omega_2^2 & -k \\ -k & k_2+k-m_2\omega_2^2 \end{pmatrix}\begin{pmatrix} X_1^2 \\ X_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let $X_1^2 = 1 \Rightarrow X_2^2 = \dfrac{k_1+k-m_1\omega_2^2}{k} = \mu_2$

so $X^2 = \begin{pmatrix} 1 \\ \mu_2 \end{pmatrix} \leftarrow$ 2nd mode shape.

Total response:

$$\underset{\sim}{x} = c_1 X^1 e^{i\omega_1 t} + c_2 X^1 e^{-i\omega_1 t} + c_3 X^2 e^{i\omega_2 t} + c_4 X^2 e^{-i\omega t}$$

This can be written

$$\underset{\sim}{x} = D_1 \underset{\sim}{X}^1 \cos\omega_1 t + D_2 \underset{\sim}{X}^1 \sin\omega_1 t + D_3 \underset{\sim}{X}^1 \cos\omega_2 t + D_4 \underset{\sim}{X}^1 \sin\omega_2 t$$

where the D's are determined from the initial conditions $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ and $\dot{x}(0) = \underset{\sim}{v}_0$

we can also write:

$$x = A \underset{\sim}{X}^1 \sin(\omega_1 t + \phi_1) + B \underset{\sim}{X}^2 \sin(\omega_2 t + \phi_2)$$

where A, B, ϕ_1 and ϕ_2 are determined from the ICs.

<u>Ex</u> Let $m_1 = m_2 = m$, $k_3 = k_2$

then $\omega_1^2 = \dfrac{k}{m}$, $\omega_2^2 = \dfrac{k_1 + 2k}{m}$ (since $\omega_{1,2}^2 = \dfrac{k_1 + k}{m} \mp \dfrac{k}{m}$)

$$\Rightarrow \quad \mu_1 = \frac{k_1 + k - k_1}{k} = 1 \quad \Rightarrow \quad X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

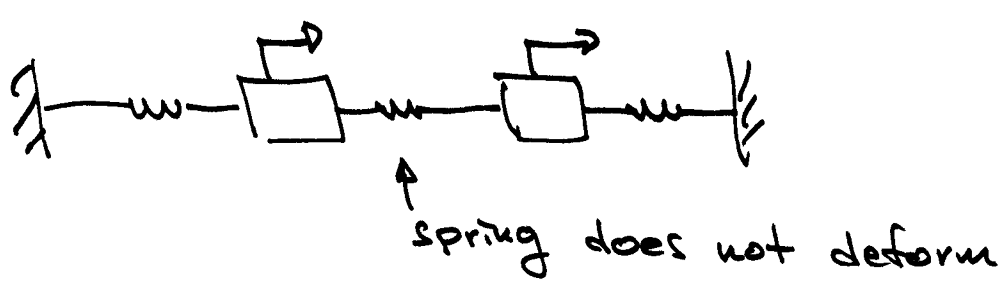
$$\mu_2 = \frac{k_1 + k - k_1 - 2k}{k} = -1 \quad \Rightarrow \quad X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

mode shapes

<u>First mode</u>

$$\underset{\sim}{x} = A \underset{\sim}{X}^1 \sin(\omega_1 t + \phi_1)$$

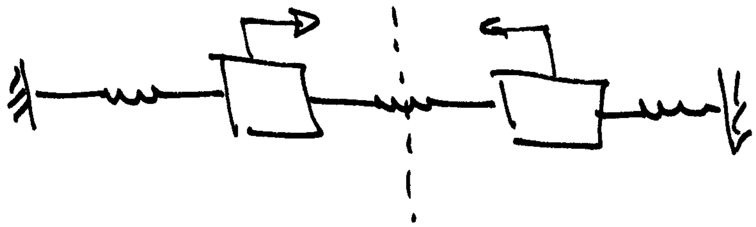
$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right)$$



2nd mode:

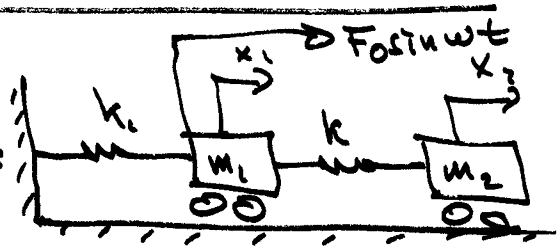
$$\underset{\sim}{x} = A \underset{\sim}{X}^2 \sin(\omega_2 t + \phi_2)$$

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin\left(\sqrt{\frac{k_1+2k}{m}} \cdot t + \phi_2\right)$$



Forced Oscillations

Consider the system:



Equ of motion:

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = F_0 \sin\omega t$$

$$m_2 \ddot{x}_2 + k_2(x_2 - x_1) = 0$$

or $M \ddot{\underset{\sim}{x}} + K \underset{\sim}{x} = \underset{\sim}{F}$

Consider the steady state osc.:

Assume

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t}$$

↑ **Not** a mode shape

Substitute into DEQ

$$(k_1+k_2-m_1\omega^2)\mathbb{X}_1 - k\mathbb{X}_2 = F_0$$

$$-k_2\mathbb{X}_1 + (k_2-m_2\omega^2)\mathbb{X}_2 = 0$$

Using Cramers Rule (or something like it)

$$\mathbb{X}_1 = \frac{(k_2-m_2\omega^2)F_0}{\Delta\omega}$$

$$\mathbb{X}_2 = \frac{k_2F_0}{\Delta\omega}$$

where $\Delta\omega = \begin{vmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{vmatrix}$

$$\Delta\omega = (\omega^2-\omega_1^2)(\omega^2-\omega_2^2)\cdot m_1 m_2$$

where ω_1, ω_2 are the natural frequencies.

For $F_0 \sin\omega t$ input, $\mathrm{Im}[F_0 e^{i\omega t}]$

then output : $\mathrm{Im}[\mathbb{X} e^{i\omega t}]$

$x_1 = \mathbb{X}_1 \sin\omega t$

$x_2 = \mathbb{X}_2 \sin\omega t$ (No phase this time since $\underline{\mathbb{X}}$ is real)

$$\bar{X}_1 = \frac{\left(\frac{k_2}{m_2} - \omega^2\right) F_0}{m_1(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}$$

$$\bar{X}_2 = \frac{\frac{k_2}{m_2} F_0}{m_1(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}$$

$$\omega_1^2 < \frac{k_2}{m_2} < \omega_2^2$$

plot amplitude and phase

$$x_1 = |\bar{X}_1| \cdot \sin(\omega t + \phi_1)$$
$$x_2 = |\bar{X}_2| \cdot \sin(\omega t + \phi_2)$$