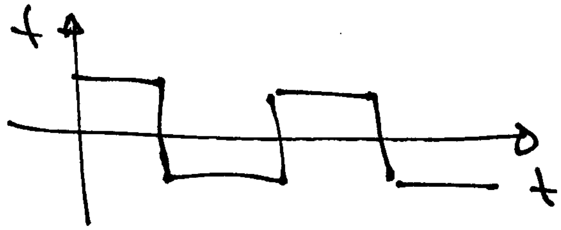
Fourier Series

We know from earlier:

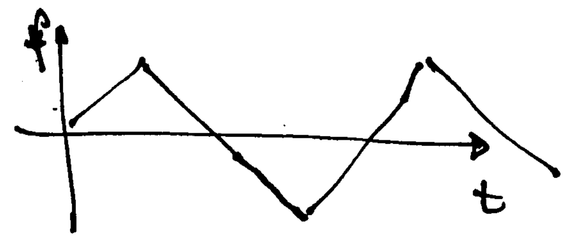
$$A\cos\omega t \longrightarrow \boxed{H(i\omega)} \longrightarrow A|H(i\omega)|\cos(\omega t+\phi)$$

But what happens when the input is not a simple sine or cosine?

eg



or



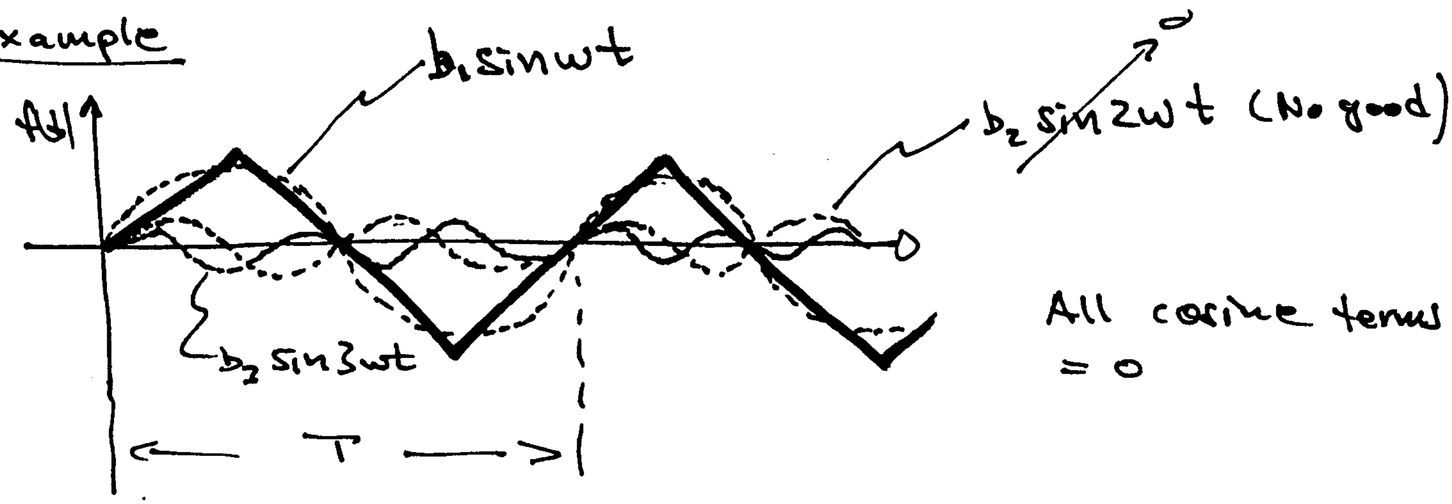
We can represent an arbitrary function of period T by a series of the form:

$$\text{input} = f(t) = \tfrac{1}{2}a_0 + a_1\cos\omega t + a_2\cos 2\omega t + a_3\cos 3\omega t + \dots$$
$$+ b_1\sin\omega t + b_2\sin 2\omega t + b_3\sin 3\omega t + \dots$$

→ where $\omega = 2\pi/T$ ←

We can now apply the previous methods to each individual term of this series in order to get the response to a general periodic disturbance.

Example



All cosine terms $= 0$

$$f(t) = b_1 \sin \omega t + \cancel{b_2 \sin 2\omega t}^{\,0} + b_3 \sin 3\omega t$$

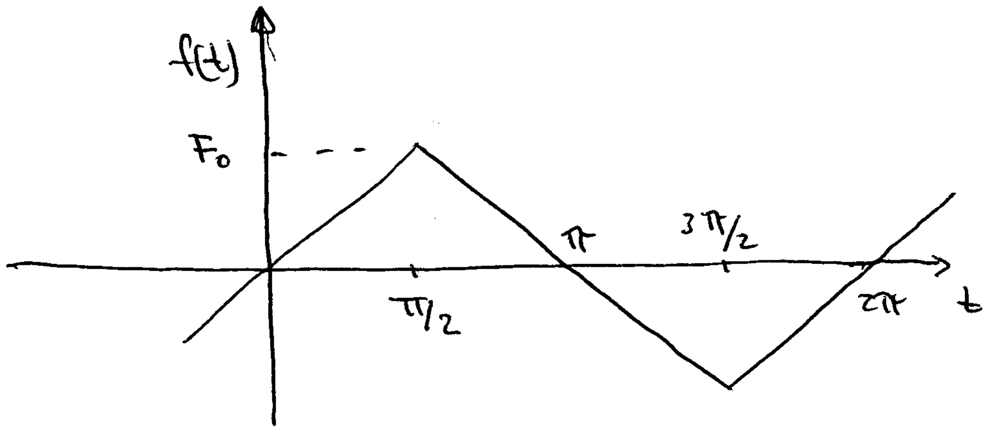
Question How do we calculate the coefficients $a_0, a_1, \ldots, b_1, b_2 \ldots$?

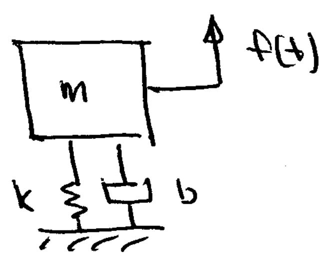
Without delay:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t) \qquad n = 0, 1, 2 \ldots$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t) \qquad n = 1, 2, 3 \ldots$$

Ex





$\omega_n = 3$

$\zeta = 0.05$

Determine the steady state response to the periodic force f(t).

We can write

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=0}^{\infty} b_n \sin(n\omega_0 t)$$

odd function so only sine terms

$\rightarrow \quad a_n = 0$

$$b_n = \frac{2}{T} \int_{-T/4}^{3T/4} f(t) \cdot \sin(n\omega_0 t) \quad , T = 2\pi \rightarrow \omega_0 = 1$$

For $-\pi/2 < t < \pi/2 \quad : \quad f(t) = \frac{2F_0}{\pi} \cdot t$

$\pi/2 < t < 3\pi/2 \quad : \quad f(t) = 2F_0 \left(1 - \frac{t}{\pi}\right)$

$$b_n = \frac{2}{2\pi}\left[\int_{-\pi/2}^{\pi/2} \frac{2F_0}{\pi}\cdot t \sin(nt)\,dt + \int_{\pi/2}^{3\pi/2} 2F_0\left(1-\frac{t}{\pi}\right)\sin(nt)\,dt\right]$$

$$b_n = \frac{8F_0}{n^2\pi^2}\sin n\pi/2$$

$$b_1 = \frac{8F_0}{\pi^2}, \quad b_2 = 0, \quad b_3 = -\frac{8F_0}{9\pi^2}, \quad b_4 = 0, \ldots$$

so

$$\boxed{f(t) = \frac{8F_0}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin(2n-1)t} \quad (1)$$

Response

We know that the response is

$x(t) = \frac{F_0'}{k}\cdot(AF)\cdot\sin(\omega t+\phi)$ if the

input is $f(t) = F_0'\sin(\omega t)$

Since the system is linear we can treat each term in (1) seperatelly. and add the response to each term to get the total response.

Here
$$AF = \frac{1}{\left[\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right]^{1/2}}$$

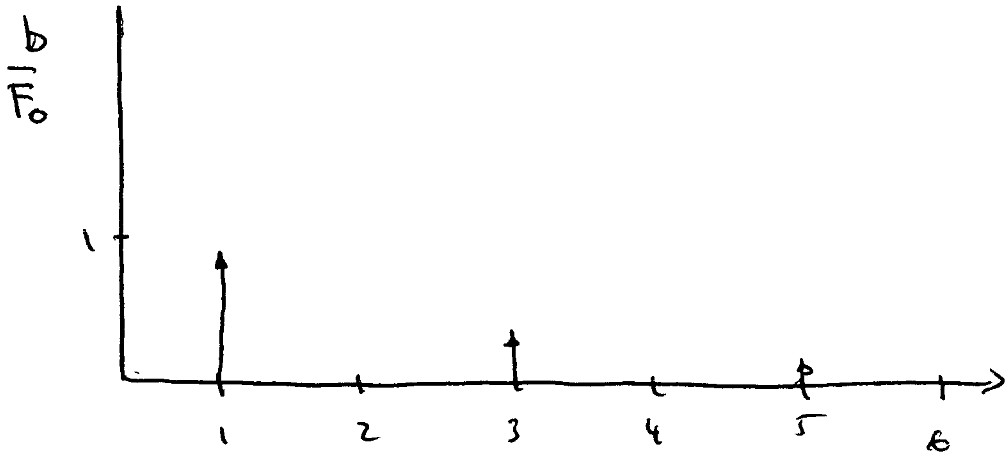
and
$$F_0' = \frac{8F_0}{\pi^2}\frac{1}{(2n-1)^2}, \quad \omega = (2n-1)\omega_0$$

so total response (steady state)

$$x = \sum_{n=1}^{\infty} \frac{\frac{8F_0}{k\pi^2}\frac{1}{(2n-1)^2}}{\left[\left[1-\left(\frac{\omega_0(2n-1)}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega_0(2n-1)}{\omega_n}\right)^2\right]^{1/2}} \cdot \sin\left((2n-1)\omega_0 t + \phi\right)$$

$$x = \sum_{n=1}^{\infty} \frac{\frac{8F_0}{k\pi^2}\frac{1}{(2n-1)^2}}{\left[\left(1-\left(\frac{2n-1}{3}\right)^2\right)^2 + 4(0.05)^2\cdot\left(\frac{2n-1}{3}\right)^2\right]^{1/2}} \sin\left((2n-1)t + \phi_n\right),$$

where $\phi_n = -\tan^{-1} \dfrac{2\cdot(0.05)\left(\frac{2n-1}{3}\right)}{1-\left(\frac{2n-1}{3}\right)^2}$


b
F0
1
1
2
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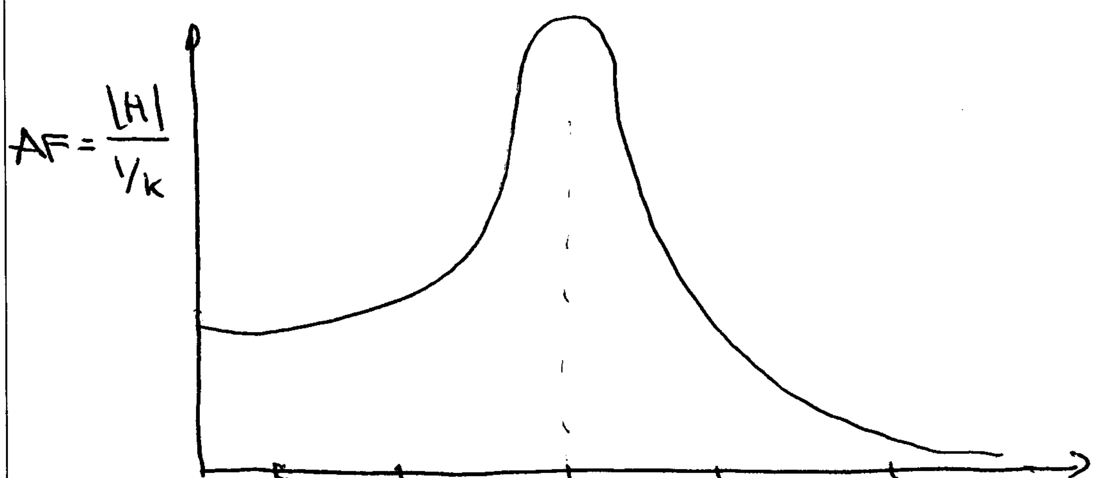
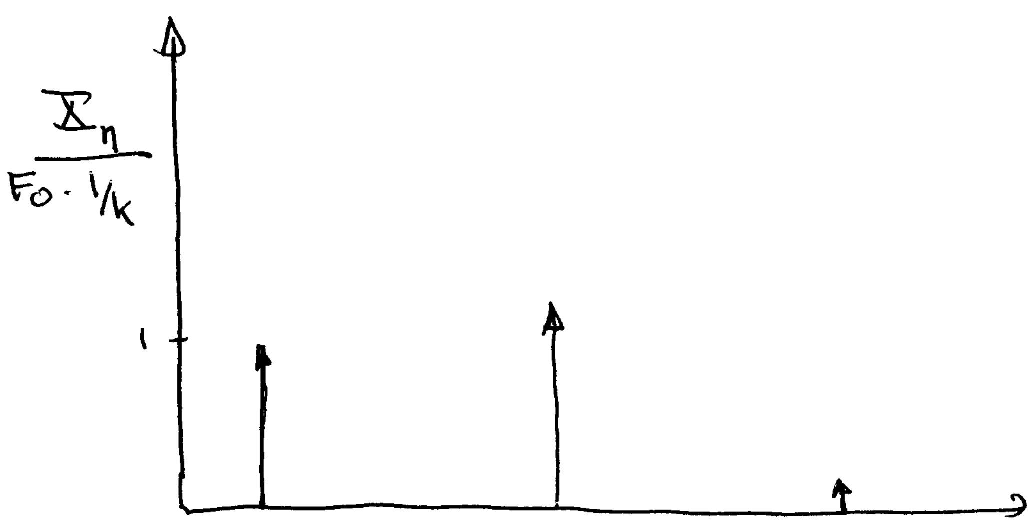

AF = |H| / 1/k


X_n / F0 · 1/k
1