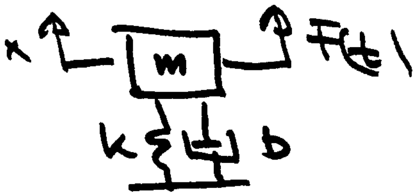
Forced Vibrations



$$\Sigma F = m\ddot{x} \Rightarrow m\ddot{x} + b\dot{x} + kx = F(t)$$

if $F(t) = F_0 \sin\omega t$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega t$$

let $\qquad = \frac{F_0}{m} e^{i\omega t}$ $\qquad \star$

Total response $\quad x_T = x_H + x_p$

where $\quad x_H$ = homogen. sol = free sol = transient

x_p = particular sol = forced sol =

= steady state soln

let $\quad x_p = C e^{i\omega t}$

plug into $\star$:

$$(-\omega^2 + 2\zeta\omega_n\omega i + \omega_n^2)\, C e^{i\omega t} = \frac{F_0}{m} e^{i\omega t}$$

$$C = \frac{F_0/m}{(\omega_n^2 - \omega^2) + i2\zeta\omega_n\omega} e^{i\omega t}$$

or

$$x_p = \frac{F_0/m}{((\omega_n^2-\omega^2)^2 + 2^2\zeta^2\omega_n^2\omega^2)^{1/2}} e^{i\omega t} \cdot e^{i\theta}$$

where $\phi = \tan^{-1} \dfrac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}$

since $\sin\omega t$ input $\rightarrow$

$$x_p = \frac{F_0/m}{[(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2]^{1/2}} \cdot \sin(\omega t - \phi)$$

steady state amplitude:

$$|x_{ss}| = \frac{F_0/m}{[(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2]^{1/2}}$$

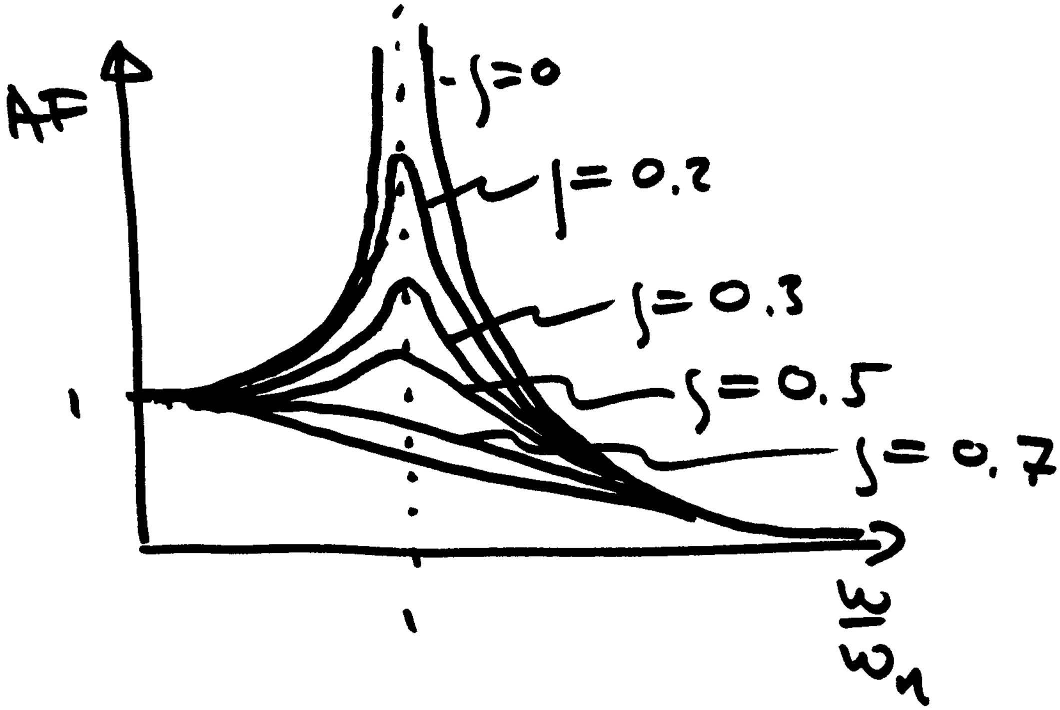
OR

$$|x_{ss}| = \frac{F_0}{k} \frac{\omega_n^2}{[(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2)^{1/2}}$$

Amplification Factor =

$$= AF = \frac{\omega_n^2}{[(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n\omega^2]^{1/2}}$$

$$= \frac{1}{\left[\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right]^{1/2}}$$



The peak response will occur at

$$\frac{dAF}{d\left(\frac{\omega}{\omega_n}\right)} = 0 \implies \frac{-\frac{1}{2}\left[2\left(1-\left(\frac{\omega}{\omega}\right)^2\right)\right]\left(-\frac{2\omega}{\omega_n}\right)+4\zeta^2\cdot 2\left(\frac{\omega}{\omega_n}\right)]}{[\qquad]^{3/2}}$$

$$\implies \omega\left[2\zeta^2-1+\left(\frac{\omega}{\omega_n}\right)^2\right]=0$$

Hence $\omega=0$ (for large ζ) or

$$2\zeta^2-1+\left(\frac{\omega}{\omega_n}\right)^2=0 \implies$$

$$\frac{\omega}{\omega_n}=\sqrt{1-2\zeta^2} \leftarrow \text{peak for small } \zeta$$

So $\boxed{\zeta < \frac{1}{\sqrt{2}}}$ for AF to have a maximum (resonance).

$$\Rightarrow \begin{cases} AF_{max} = \dfrac{1}{2\zeta(1-\zeta^2)^{1/2}} \\ \phi = \tan^{-1} 1-2\zeta^2 \end{cases}$$

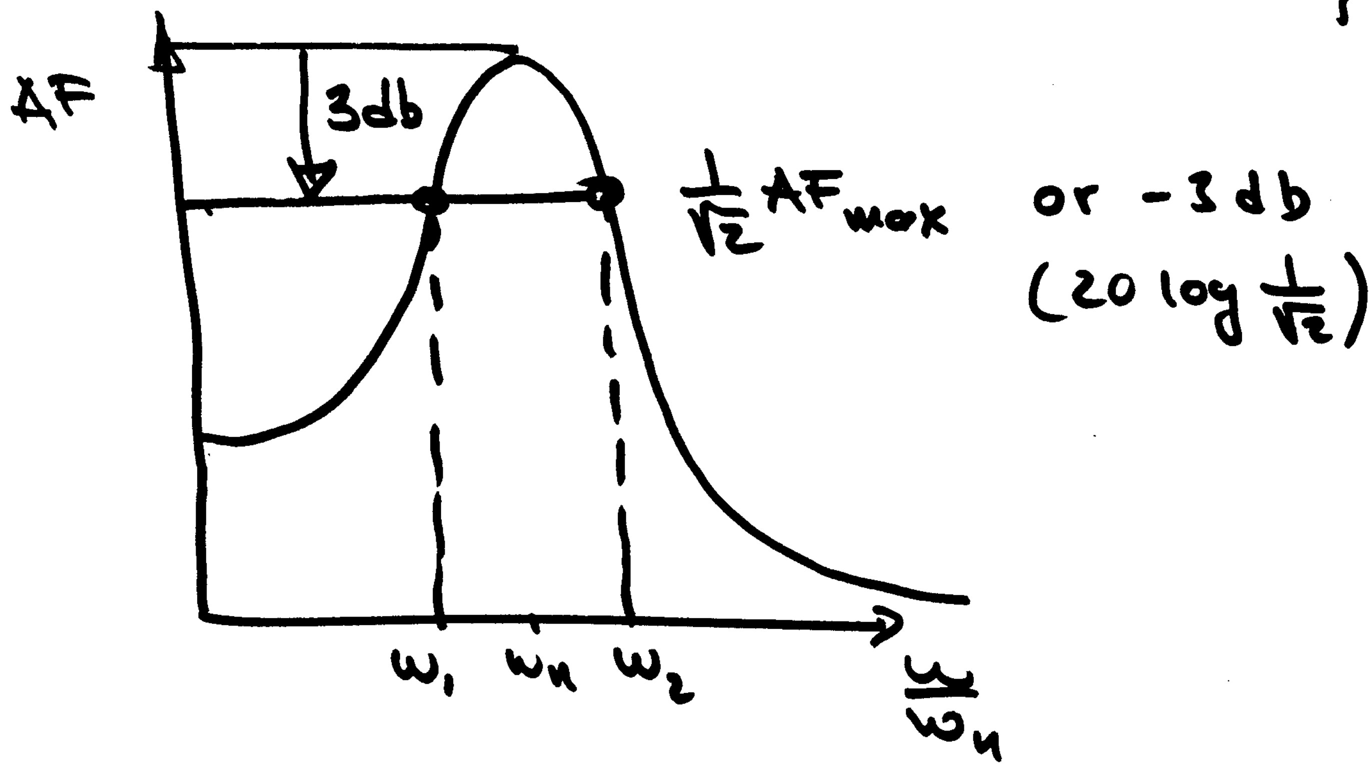
if $\zeta \ll 1$ then

$\boxed{AF_{max} \cong \frac{1}{2\zeta}}$ and $\phi \cong \tan^{-1} \frac{1}{\zeta}$

The amplification factor is $\sim \frac{1}{2\zeta}$

This is also called the

Q factor $\boxed{Q = \frac{1}{2\zeta}}$ $\zeta \ll 1$



$\omega_{1,2} = (1 \pm \zeta)\omega_n$ ← from AF and $\zeta \ll 1$

$\Rightarrow \quad \omega_2 - \omega_1 = 2\zeta\omega_n \quad \Rightarrow \quad \boxed{\zeta = \frac{\omega_2 - \omega_1}{2\omega_n}}$

or $Q = \frac{1}{2\zeta} \quad \Rightarrow \quad \boxed{Q = \frac{\omega_n}{\omega_2 - \omega_1}}$

So ζ and Q can be determined from the AF vs. $\frac{\omega}{\omega_n}$ plot.

Non Viscous Damping

$$\boxed{\text{Real Damping Mechanism}} \xrightarrow{\text{Replace with}} \boxed{\text{Viscous Damping}}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\boxed{\begin{array}{c}\Delta W_d \\ \text{energy dissipation/cycle}\end{array}} = \boxed{\begin{array}{c}\Delta W_d \\ \text{energy dissipation/cycle}\end{array}}$$

For viscous damping:

$$\Delta W_d = \text{energy dissipation/cycle} = \oint F_b \cdot dx = \int_{t_1}^{t_1+2\pi/\omega} F \frac{dx}{dt} \cdot dt$$

$$\Delta W_d = \int_0^{2\pi/\omega} b\, \dot{x}^2 dt = \int_0^{2\pi/\omega} A^2 \omega^2 \cos^2(\omega t - \phi)\, dt$$

$$\boxed{\Delta W_d = b A^2 \omega \pi}$$

For non-viscous damping:

Want the "equivalent viscous damper" to dissipate the input energy (at steady state)

Input : $\Delta W = \oint F\cdot dx = \int_0^{2\pi/\omega} F\cdot \frac{dx}{dt}\cdot dt$

$$\Delta W = \int_0^{2\pi/\omega} F_0\cdot \sin\omega t\cdot A\omega\cos(\omega t-\phi)\,dt = F_0\cdot A\omega\sin\phi\left(\frac{\pi}{\omega}\right)$$

$$= \pi\cdot F_0\cdot A\sin\phi$$

at resonance :

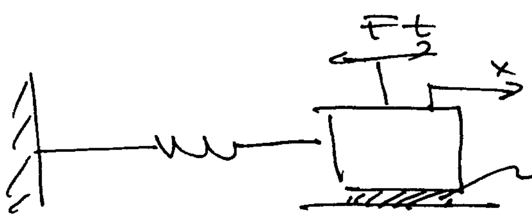
$$\Delta W \approx \pi\cdot F_0\cdot A_{res}$$

so

$$b_{eq}A_{res}^2\,\omega_{res}\,\pi = \pi\cdot F_0\,A_{res}$$

$$\boxed{b_{eq} = \frac{\pi\cdot F_0}{A_{res}\cdot\omega_{res}\cdot\pi}}$$

Ex



coulomb damping
ie friction