Problem 15–3

From the figure the waveform is periodic with period T, and may be written:

$$f(t) = 1 - 4\,|t|\,/T \qquad\qquad |t| < T/2$$

The sinusoidal form of the Fourier series is found in two parts from Table 15.1:

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)dt \\ &= \frac{2}{T}\int_{-T/2}^{0}\left(1+\frac{4t}{T}\right)\cos(2\pi nt/T)dt + \frac{2}{T}\int_{0}^{T/2}\left(1-\frac{4t}{T}\right)\cos(2\pi nt/T)dt \end{aligned}$$

Noting that

$$\int t\cos(at)dt = \frac{1}{a^2}\left(\cos(at) + at\sin(a)\right)$$

$$\begin{aligned} a_n &= \frac{2}{T}\left[\frac{T}{2\pi n}\sin\frac{2\pi n}{T}t + \frac{4}{T}\frac{T^2}{(2\pi n)^2}\left(\cos\frac{2\pi n}{T}t + \frac{2\pi n}{T}t\sin\frac{2\pi n}{T}t\right)\right]_{-T/2}^{0} \\ &\quad + \frac{2}{T}\left[\frac{T}{2\pi n}\sin\frac{2\pi n}{T}t + \frac{4}{T}\frac{T^2}{(2\pi n)^2}\left(\cos\frac{2\pi n}{T}t - \frac{2\pi n}{T}t\sin\frac{2\pi n}{T}t\right)\right]_{-T/2}^{0} \\ &= \frac{4\left(1-\cos(n\pi)\right)}{n^2\pi^2} \end{aligned}$$

Similarly

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)dt \\ &= \frac{2}{T}\int_{-T/2}^{0}\left(1+\frac{4t}{T}\right)\sin(2\pi nt/T)dt + \frac{2}{T}\int_{0}^{T/2}\left(1-\frac{4t}{T}\right)\sin(2\pi nt/T)dt \\ &= 0 \end{aligned}$$

Thus $f(t)$ is represented by a cosine series, but in addition we note that $a_n = 0$ when n is even, and $a_n = 8/\pi^2 n^2$ when n is odd. Therefore

$$f(t) = \sum_{n=1}^{\infty} \frac{8}{\pi^2(2n-1)^2}\cos\frac{2(2n-1)\pi}{T}t$$

The exponential series may be found by writing the above expression in terms of exponentials

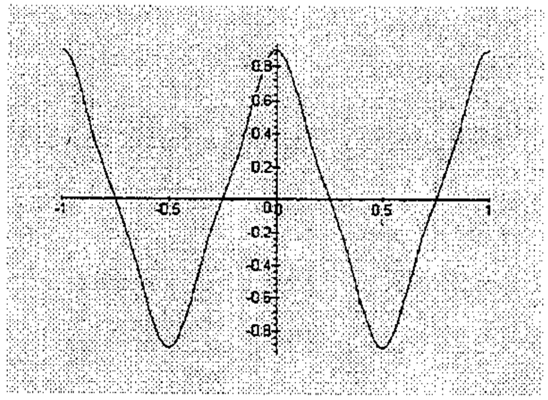
$$f(t) = \sum_{n=1}^{\infty} \frac{8}{\pi^2(2n-1)^2}\frac{1}{2}\left(e^{j2\pi(2n-1)t/T} + e^{j2\pi(2n-1)t/T}\right)$$

so that

$$F_n = F_{-n} = a_n/2 = \frac{2\left(1-\cos(n\pi)\right)}{n^2\pi^2}$$

The series with the first four terms is

$$\begin{aligned} f(t) &= a_0 + a_1\cos(2\pi t/T) + a_2\cos(4\pi t/T) + a_3\cos(6\pi t/T) \\ &= 0 + 8/\pi^2\cos(2\pi t/T) + 0 + 8/(9\pi^2)\cos(6\pi t/T) \end{aligned}$$



Problem 15–4

From Fig. 15.25 the period is $T = 0.04$ seconds. The complex Fourier series coefficients are (Table 15.1):

$$\begin{aligned} F_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_{-T/4}^{T/4}\cos(2\pi t/T)e^{-j2\pi nt/T}dt \\ &= \frac{1}{2T}\int_{-T/4}^{T/4}\left(e^{j2\pi t/T} + e^{-j2\pi t/T}\right)e^{-j2\pi nt/T}dt \\ &= \frac{1}{2T}\int_{-T/4}^{T/4}\left(e^{-j2\pi t(n-1)/T} + e^{-j2\pi t(n+1)/T}\right)dt \\ &= \frac{j}{4\pi}\left[\frac{1}{n-1}e^{-j2\pi(n-1)t/T} + \frac{1}{n+1}e^{-j2\pi(n+1)t/T}\right]_{-T/4}^{T/4} \\ &= \frac{1}{2\pi}\left[\frac{\sin(\pi(n-1)/2)}{n-1} + \frac{\sin(\pi(n+1)/2)}{n+1}\right] \end{aligned}$$

The coefficients are real, indicating a cosine function:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t))$$

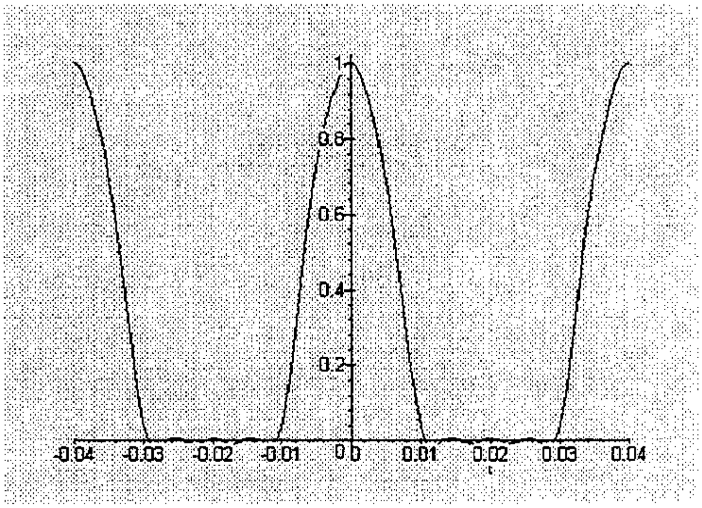
where (Eq. 15.10)

$$\begin{aligned} a_n &= F_n + F_{-n} = \frac{1}{\pi}\left[\frac{\sin(\pi(n-1)/2)}{n-1} + \frac{\sin(\pi(n+1)/2)}{n+1}\right] \\ b_n &= j(F_n - F_{-n}) = 0 \end{aligned}$$

For $T = 0.04$ s, the first few terms of the series are:

$$f(t) = 0.318 + 0.5\cos(50\pi t) + 0.212\cos(100\pi t) - 0.0424\cos(200\pi t) + 0.01818\cos(300\pi t) + \ldots$$

The sum of these terms is shown below



The average (or dc) value is

$$\overline{f(t)} = F_0 = \frac{1}{2\pi}\left[\frac{\sin(-\pi/2)}{-1} + \frac{\sin(\pi/2)}{1}\right] = 0.318$$

Problem 15–5

From Table 15.1

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j2\pi nt/T)}dt$$

(a) $\hat{f}(t) = f(-t) \quad \Longleftrightarrow \quad \hat{F}_n = \frac{1}{T}\int_{-T/2}^{T/2} f(-t)e^{-j2\pi nt/T)}dt$

Change the variable of integration $\nu = -t$:

$$\hat{F}_n = \frac{1}{T}\int_{-T/2}^{T/2} f(\nu)e^{j2\pi n\nu/T)}d\nu = F_{-n}$$

(b) $\hat{f}(t) = \frac{df}{dt} \quad \Longleftrightarrow \quad \hat{F}_n = \frac{1}{T}\int_{-T/2}^{T/2} \frac{df}{dt}e^{-j2\pi nt/T)}dt$

Integrate by parts

$$\begin{aligned}\hat{F}_n &= \frac{1}{T}\left[f(t)e^{-j2\pi nt/T}\right]_{-T/2}^{T/2} + \int_{-T/2}^{T/2}\frac{j2\pi n}{T}f(t)e^{-j2\pi nt/T)}dt \\ &= \frac{j2\pi n}{T}F_n\end{aligned}$$

(c) $\hat{f}(t) = f(t-T) \quad \Longleftrightarrow \quad \hat{F}_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t-T)e^{-j2\pi nt/T)}dt$

Since $f(t-T) = f(t)$, it follows that $\hat{F}_n = F_n$.

(d) $\hat{f}(t) = \int_{-\infty}^{t} f(t)dt \quad$ (Assume $F_0 = 0$.)

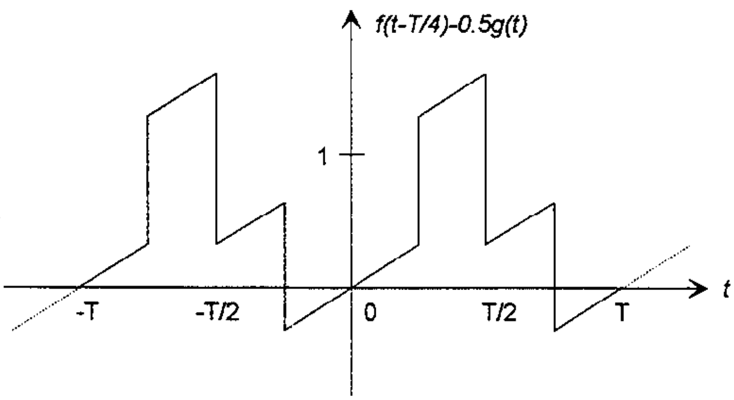
Since $f(t) = \frac{d\hat{f}}{dt}$ from (b) above

$$F_n = \frac{j2\pi n}{T}\hat{F}_n$$
$$\hat{F}_n = -\frac{jT}{2\pi n}F_n$$

Problem 15–6

Let the waveform in Fig. 15.4 be designated $f(t)$ and the waveform in Fig. 15.5 be designated $g(t)$, each with period T.

(a) The waveform $r(t) = f(t-T/4) + 0.5g(t)$ is shown below



(b) From Example 15.4 the Fourier series for $f(t)$ is

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left[\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t) + \frac{1}{7}\sin(7\omega_0 t) + \ldots\right]$$

When the function is delayed by a time $\tau = T/4$, the Fourier series is modified by a phase shift $n\omega_0\tau = n\pi/2$. The Fourier series for $g(t)$ is (from Example 15.5)

$$g(t) = \frac{2}{\pi}\left[\sin(\omega_0 t) - \frac{1}{2}\sin(2\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) - \frac{1}{4}\sin(4\omega_0 t) + \ldots\right]$$

The sum $r(t) = f(t-T/4) + 0.5g(t)$ is therefore

$$r(t) \;=\; \frac{1}{2} + \frac{2}{\pi}\left[\frac{1}{2}\sin(\omega_0 t) + \sin(\omega_0 t - \pi/2) - \frac{1}{4}\sin(2\omega_0 t) + \frac{1}{6}\sin(3\omega_0 t)\right.$$

$$
\begin{aligned}
&+\frac{1}{3}\sin(3\omega_0 t - 3\pi/2) - \frac{1}{8}\sin(4\omega_0 t + \frac{1}{10}\sin(5\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t - 5\pi/2) + \\
&-\frac{1}{12}\sin(6\omega_0 t) + \frac{1}{14}\sin(7\omega_0 t) + \frac{1}{7}\sin(7\omega_0 t - 7\pi/2) + \ldots\Big] \\
= \;& \frac{1}{2} + \frac{2}{\pi}\Big[\frac{1}{2}\sin(\omega_0 t) - \cos(\omega_0 t) - \frac{1}{4}\sin(2\omega_0 t) + \frac{1}{6}\sin(3\omega_0 t) \\
&+\frac{1}{3}\cos(3\omega_0 t) - \frac{1}{8}\sin(4\omega_0 t) + \frac{1}{10}\sin(5\omega_0 t) - \frac{1}{5}\cos(5\omega_0 t) \\
&-\frac{1}{12}\sin(6\omega_0 t) + \frac{1}{14}\sin(7\omega_0 t) + \frac{1}{7}\cos(7\omega_0 t) + \ldots\Big]
\end{aligned}
$$

Problem 15–7

(a) The Fourier series for $f(t) = |\sin(\omega t)|$ is defined from

$$F_n = \frac{1}{T}\int_{-T/2}^{0} -\sin(2\pi t/T)e^{-j2\pi nt/T}dt + \frac{1}{T}\int_{0}^{+T/2} \sin(2\pi t/T)e^{-j2\pi nt/T}dt$$

where $T = 2\pi/\omega$, and which may be evaluated in a similar manner to Problem 15.4 to give

$$
\begin{aligned}
F_n &= -\frac{\cos(n\pi)+1}{\pi(n^2-1)} \\
&= \begin{cases} -\dfrac{2}{\pi(n^2-1)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}
\end{aligned}
$$

The trigonometric series is found from Eq. 15.10, that is

$$
\begin{aligned}
a_n &= F_n + F_{-n} = -4/\pi(n^2-1) \qquad (n \text{ even}) \\
b_n &= 0
\end{aligned}
$$

The series is

$$
\begin{aligned}
f(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(n\omega t) \\
&= 0.63663 - 0.4244\cos(2\omega t) - 0.0848\cos(4\omega t) - 0.0364\cos(6\omega t) \\
&\quad -0.0202\cos(8\omega t) - 0.0128\cos(10\omega t) - 0.0089\cos(12\pi t) + \ldots
\end{aligned}
$$

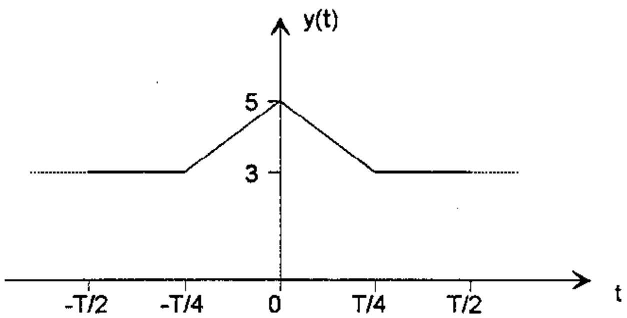
from which the spectrum may be plotted.

(b) The frequency response is

$$H(j\omega) = \frac{1}{j0.2\omega + 1}$$

Problem 15–9

At a rotational speed of 100 rpm, the angular velocity is $200\pi/60$ rad/s, and the period is $T = 0.6$ s. Then $y(t)$ is as shown below:



(a) The Fourier series is found in a piecewise manner:

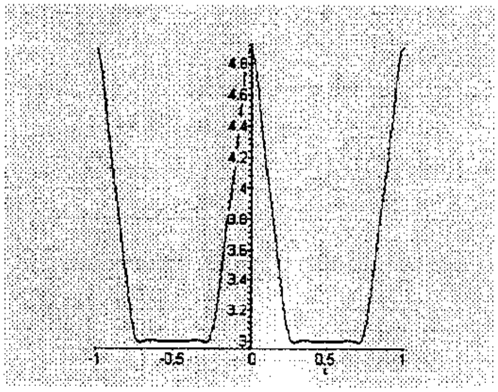
$$\begin{aligned} y(t) &= 5 + \frac{8}{T}t \qquad -T/4 \le t < 0 \\ &= 5 - \frac{8}{T}t \qquad 0 \le t < T/4 \\ &= 3 \qquad T/4 \le t < 3T/4 \end{aligned}$$

Working directly in the trigonometric series, noting that y(t) is an even function ($b_n = 0$):

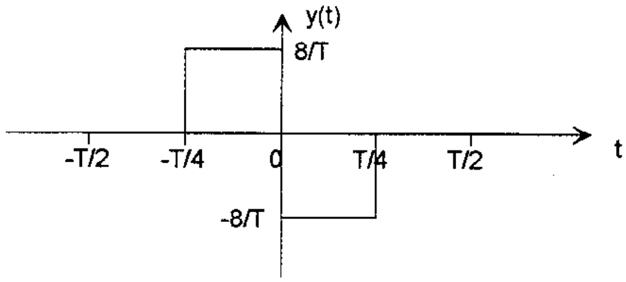
$$\begin{aligned} a_n &= \frac{2}{T}\int_{-T/4}^{0}\left(5 + \frac{8}{T}t\right)\cos(2\pi nt/T)dt + \frac{2}{T}\int_{0}^{T/4}\left(5 - \frac{8}{T}t\right)\cos(2\pi nt/T)dt \\ &\quad + \frac{2}{T}\int_{T/4}^{3T/4} 3\cos(2\pi nt/T)dt \\ &= \frac{3n\pi\sin(n\pi/2) - 4\cos(n\pi/2) + 4}{n^2\pi^2} + \frac{3n\pi\sin(n\pi/2) - 4\cos(n\pi/2) + 4}{n^2\pi^2} \\ &\quad + \frac{3\sin(3n\pi/2) - 3\sin(n\pi/2)}{n\pi} \\ &= \begin{cases} 7 & n = 0 \\ \dfrac{8\,(1 - cos(n\pi/2))}{n^2\pi^2} & n = 1, 2, 3\ldots \end{cases} \end{aligned}$$

where the case for $n = 0$ must be computed using l'Hospital's rule.

The function formed by the first eight terms is shown below:



(b) The linear velocity of the follower is $v(t) = dy/dt$, that is:



and the function is odd and therefore represented by sine terms only:

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-T/4}^{0}\left(\frac{8}{T}\right)\sin(2\pi nt/T)dt + \frac{2}{T}\int_{0}^{-T/4}\left(-\frac{8}{T}\right)\sin(2\pi nt/T)dt \\ &= \frac{16(\cos(n\pi/2)-1)}{n\pi T} \end{aligned}$$

Problem 15–10

$$H(s) = \frac{40000}{s^2 + 20s + 40000}$$

then for the system $\omega_n = 200$ rad/s, and $\zeta = 0.05$ so that the system is very lightly damped and will exhibit resonant behavior if excited at frequencies near to ω_n.

(a) In normal operation with a nominal shaft speed of 50 rad/s

$$\Omega(t) = 50\left(1 + \frac{1}{4}\sin(100t) + \frac{1}{8}\sin(200t)\right)$$

and the fourth harmonic component (angular frequency of 200 rad/s) will excite the resonance i the shaft.

(b) When the nominal shaft speed is increased to 100 rad/s,

$$\Omega(t) = 100\left(1 + \frac{1}{4}\sin(200t) + \frac{1}{8}\sin(400t)\right)$$

and now the second harmonic is at the resonant frequency, therefore significant vibrations would be expected.

Problem 15–11

(a) Write the sine in its exponential form $\sin(2\pi t/T) = (e^{j2\pi t/T} - e^{-j2\pi t/T})/2j$, then

$$\begin{aligned} F(j\omega) &= \frac{1}{2j}\int_{-T/2}^{T/2}\left(e^{j2\pi t/T} - e^{-j2\pi t/T}\right)e^{-j\omega t}dt \\ &= \frac{T/2}{2\pi - \omega T}\left[e^{-j(\omega-2\pi/T)t}\right]_{-T/2}^{T/2} + \frac{T/2}{2\pi + \omega T}\left[e^{-j(\omega+2\pi/T)t}\right]_{-T/2}^{T/2} \\ &= \frac{j4\pi T\sin(\omega T/2)}{\omega^2T^2 - 4\pi^2} \end{aligned}$$

(b) Write the cosine in its exponential form $\cos(2\pi t/T) = (e^{j2\pi t/T} + e^{-j2\pi t/T})/2$, then

$$\begin{aligned} F(j\omega) &= \frac{1}{2}\int_{-T/4}^{T/4}\left(e^{j2\pi t/T} + e^{-j2\pi t/T}\right)e^{-j\omega t}dt \\ &= \frac{jT/2}{2\pi - \omega T}\left[e^{-j(\omega-2\pi/T)t}\right]_{-T/4}^{T/4} + \frac{jT/2}{2\pi + \omega T}\left[e^{-j(\omega+2\pi/T)t}\right]_{-T/4}^{T/4} \\ &= \frac{-4\pi T\cos(\omega T/4)}{\omega^2T^2 - 4\pi^2} \end{aligned}$$

(c) Write the expression in its exponential form $e^{-2t}\cos(2\pi t) = (e^{-2+j2\pi t} + e^{-2-j2\pi t})/2$, then

$$\begin{aligned} F(j\omega) &= \frac{1}{2}\int_{0}^{\infty}\left(e^{-2+j2\pi t/T} + e^{-2-j2\pi t/T}\right)e^{-j\omega t}dt \\ &= \frac{1/2}{-2 - j(\omega - 2\pi)}\left[e^{-2-j(\omega-2\pi)t}\right]_0^{\infty} + \frac{1/2}{-2 - j(\omega + 2\pi)}\left[e^{-2-j(\omega+2\pi)t}\right]_0^{\infty} \\ &= \frac{1}{2}\left[\frac{1}{2 + j(\omega - 2\pi)} + \frac{1}{2 + j(\omega + 2\pi)}\right] \\ &= \frac{2 + j\omega}{4 + \pi^2 - \omega^2 + j4\omega} \end{aligned}$$

Problem 15–12

(a) The waveform in Fig. 15.28(a) is:

$$\begin{aligned} f_a(t) &= \frac{2}{T}t & 0 \le t < T/2 \\ &= \left(2 - \frac{2}{T}t\right) & T/2 \le t < T \end{aligned}$$