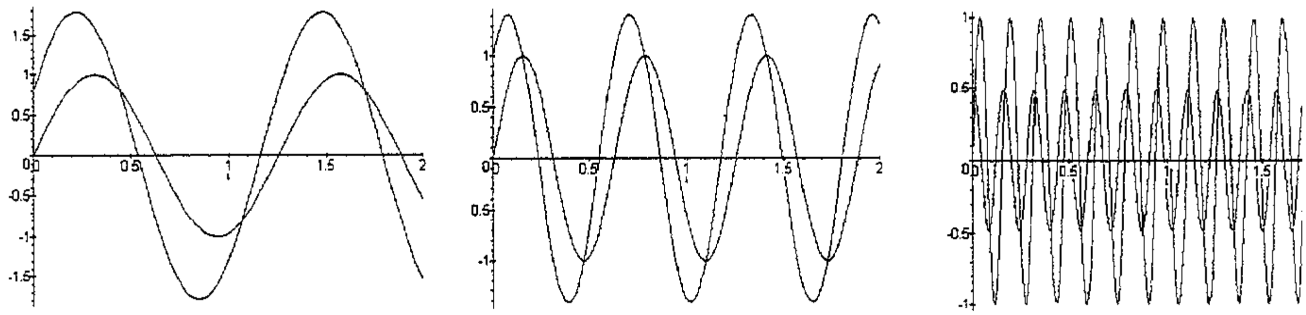
. The three wavefrorms are shown below:




(c) As the input frequency increases (i) the output amplitude decreases, and (ii) the phase shift becomes more negative.

**Problem 14.2**

For the circuit in Figure 14.24(a)

(a) The transfer function is

$$H(s) = \frac{1}{RCs+1}$$

(b) The frequency response is

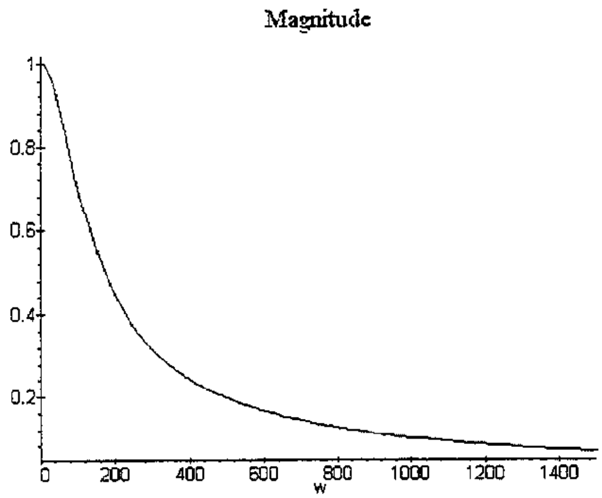
$$H(j\omega) = H(s)|_{s=j\omega} = \frac{1}{jRC\omega+1}$$

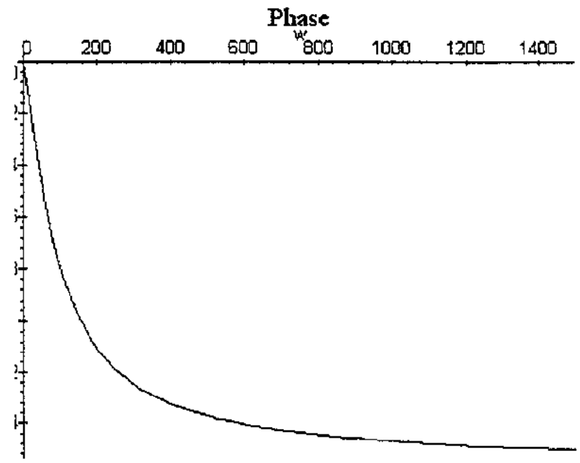
giving

$$\begin{aligned} |H(j\omega)| &= \left|\frac{1}{jRC\omega+1}\right| = \frac{1}{\sqrt{(RC1\omega)^2+1}} \\ \angle H(j\omega) &= \angle(1) - \angle(jRC\omega+1) = \tan^{-1}(RC\omega) \end{aligned}$$

The amplitude and phase plots are shown on the next page

(c) This is a low-pass system begause high frequencies are attenuated.




For the circuit in Figure 14.24(b)

**(a)** The transfer function is

$$H(s) = \frac{RCs}{RCs + 1}$$

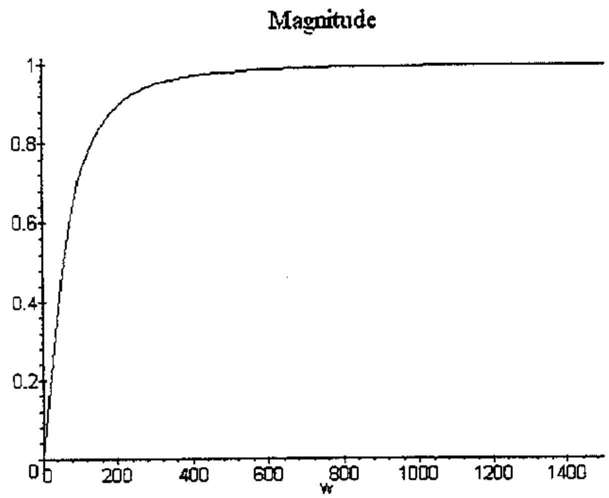
**(b)** The frequency response is

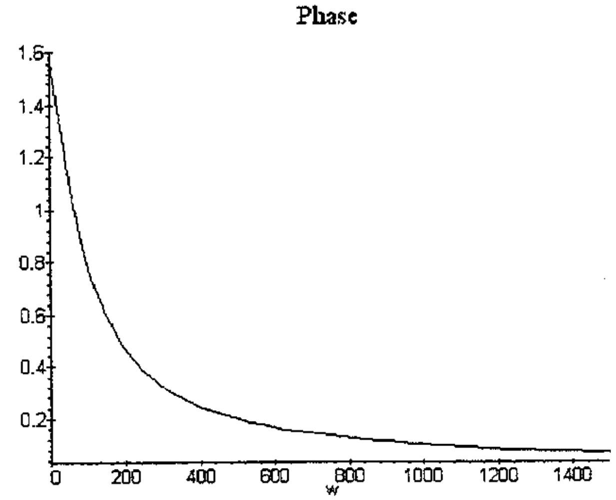
$$H(j\omega) = H(s)|_{s=j\omega} = \frac{jRC\omega}{jRC\omega + 1}$$

giving

$$\begin{aligned}
|H(j\omega)| &= \left|\frac{jRC\omega}{jRC\omega + 1}\right| = \frac{RC\omega}{\sqrt{(RC\omega)^2 + 1}} \\
\angle H(j\omega) &= \angle(jRC\omega) - \angle(jRC\omega + 1) = \pi/2 - \tan^{-1}(RC\omega)
\end{aligned}$$

The amplitude and phase plots are shown below:




**(c)** This is a high-pass system begause as the frequency approaches zero $|H(j\omega)|$ tends to zero.

## Problem 14.3

For the Kelvin model shown In Figure 14.25(a)

**(a)** The system transfer function is

$$H(s) = \frac{s}{Bs + K}$$

**(b)** The frequency response is

$$H(j\omega) = \frac{j\omega}{jB\omega + K} = \frac{j0.01\omega}{j0.2\omega + 1}$$

with $K = 100$ N/m, and $B = 20$ N-s/m.

**(c)** The frequency response magnitude function is

$$|H(j\omega)| = \frac{0.01\omega}{\sqrt{(0.2\omega)^2 + 1}}$$

This sytem will not respond to low frequencies because of the zero at the origin. At high frequencies the response will asymptotically aproach a constant value.

For the Maxwell model shown In Figure 14.25(b)

**(a)** The system transfer function is

$$H(s) = \frac{V(s)}{F(s)} = \frac{1}{K}s + \frac{1}{B}$$

(b) The frequency response is

$$H(j\omega) = j\frac{1}{K}\omega + \frac{1}{B} = j0.01\omega + 0.05$$

with $K = 100$ N/m, and $B = 20$ N-s/m.

(c) The frequency response magnitude function is

$$|H(j\omega)| = \sqrt{(0.01\omega)^2 + 0.05^2}$$

At low frequencies the response will approach a constant value ( $|H(j\omega)| \to 0.05$), while at high frequencies the response magnitude will increase without bound.

Note: Both of these models ignore inertial forces.

**Problem 14.4**

$$H(s) = \frac{T(s)}{Q_s} = \frac{1/(mc_p)}{s + (hA/(mc_p))}$$

(a) From the transfer function:

$$\frac{dT}{dt} + \frac{hA}{mc_p}T = \frac{1}{mc_p}Q_s$$

(b) The steady-state temperature $T_{ss}$ is found by letting all derivatives approach zero, that isIf the solar heat flow is a constant, that is

$$T_{ss} = \frac{1}{hA}Q_o$$

(c) From the transfer function:

$$H(j\omega) = \frac{T(j\omega)}{Q_s(j\omega)} = \frac{1/(mc_p)}{j\omega + (hA/(mc_p))}$$

Then

$$|H(j\omega)| = \frac{1/(mc_p)}{\sqrt{\omega^2 + (hA/(mc_p))^2}}, \qquad \angle H(j\omega) = -\tan^{-1}\frac{mc_p}{hA}$$

(d) If $Q_s(t) = Q_o \sin(\omega t - \pi/2) + Q_{avg}$ and $\omega = 2\pi/365$ rad/day, the annual fluctuation of the pond temperature $\Delta(T)$ is

$$\Delta(t) = Q_o \left|H(j\omega)\right|_{\omega=2\pi} = \frac{1/(mc_p)}{\sqrt{(2\pi)^2 + (hA/(mc_p))^2}}$$

(e) The annual fluctuation of pond temperature is descibed by

$$T(t) = T_{avg} + \frac{1/(mc_p)}{\sqrt{(2\pi)^2 + (hA/(mc_p))^2}} \sin\left(\omega t - \pi/2 - \tan^{-1}\frac{mc_p}{hA}\right)$$

which is a maximum when $\omega t - \pi/2 - \tan^{-1}(mc_p/hA) = \pi/2$ or

$$\begin{aligned} t_{max} &= (\pi/2 - (-\pi/2 - \tan^{-1} mc_p/hA))/2\pi \text{ years} \\ &= 365\left(\frac{1}{2} + \frac{1}{2\pi}\tan^{-1}\frac{mc_p}{hA}\right) \text{ days from the start of the year.} \end{aligned}$$

**Problem 14.5**

(a) We do not derive the transfer function in detail here. Either the linear graph or impedance methods may be used to show that the transfer function between the *velocity* of the mass and the *velocity* of the base is :

$$\frac{V_m(s)}{V_b(s)} = \frac{Bs + K}{ms^2 + Bs + K}$$

We note that the displacements of the mass ($x(t)$) and the base ($u(t)$) are the integrals of the respective velocities, and therefore

$$\frac{X(s)}{U(s)} = \frac{Bs + K}{ms^2 + Bs + K}$$

Also, $y(t) = x(t) - u(t)$ so that

$$H(s) = \frac{Y(s)}{U(s)} = \frac{X(s)}{Us} - 1 = \frac{s^2}{s^2 + (B/m)s + K/m}$$

(b)

$$H(j\omega) = H(s)|_{s=j\omega} = \frac{m(j\omega)^2}{m(j\omega)^2 + jB\omega + K}$$

and

$$|H(j\omega)| = \frac{m\omega^2}{\sqrt{((K/m - \omega^2) + (B/m)^2}}$$

$$\angle H(j\omega) = \pi - tan^{-1}\left(\frac{(B/m)\omega}{K/m - \omega^2}\right)$$

(c) Consider the behavior of $H(j\omega)$ when

(i) $\omega \ll K/m$. In this case

$$H(j\omega) \approx \frac{(j\omega)^2}{(K/m)}$$

which is the frequency response of the differerential equation

$$y(t) = \frac{K}{m}\frac{d^2u}{dt^2}$$

so that the seismometer acts to record the acceleration of the base.

(ii) $\omega \gg K/m$. In this case the second-order term in the denominator dominates and

$$H(j\omega) \approx 1$$

The output then follows the displacement of the base.

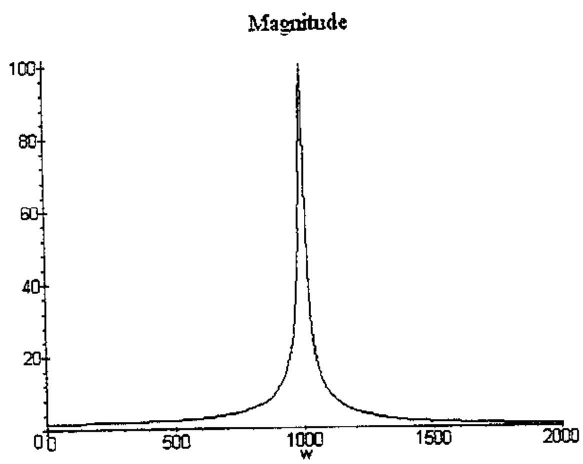
**Problem 14.6**

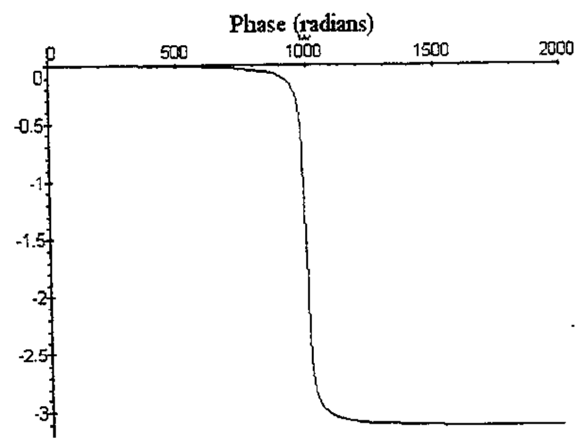
(a)

$$H(j\omega) = \frac{2 \times 10^6}{(j\omega)^2 + j20\omega + 10^6}$$

and

$$|H(j\omega)| = \frac{2 \times 10^6}{\sqrt{(10^6 - (\omega)^2)^2 + (20\omega)^2}} \quad \text{and} \quad \angle H(j\omega) = -tan^{-1}\frac{20\omega}{10^6 - \omega^2}$$




(b) When $\omega = 250$ rad/s, the amplitude and phase are

$$|H(j250)| = \frac{2 \times 10^6}{\sqrt{(10^6 - (250)^2)^2 + (20 \times 250)^2}} = 2.133$$

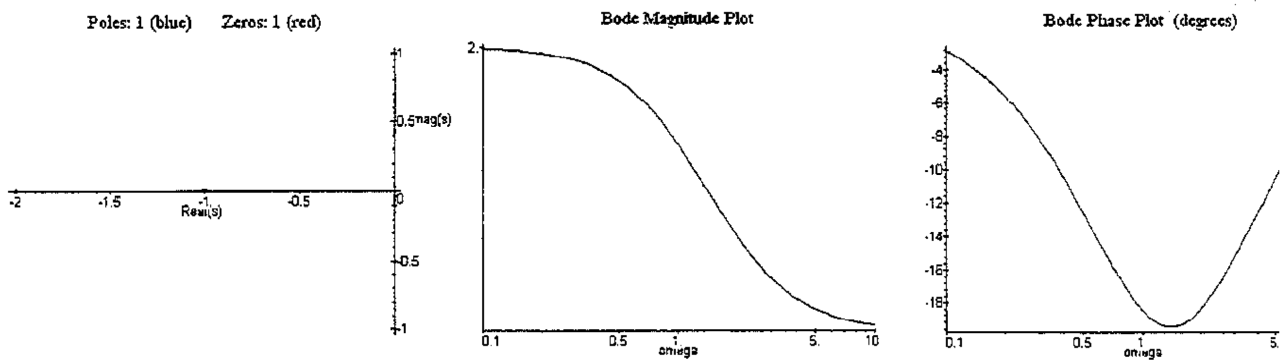
$$\angle H(j250) = -tan^{-1}\frac{20 \times 250}{10^6 - 250^2} = -.00533 \text{ rad (or -0.305°)}$$

## Problem 14.7

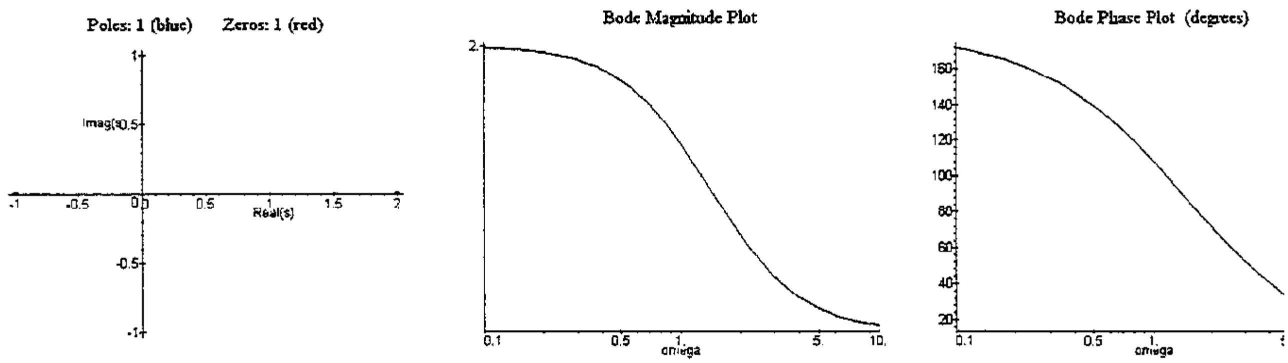
(a) No, the stability of a system is not affected by its zeros.

(b) The form of the Bode plots depends on the relative location of the pole and zero. For example below we take $|a| = 2$, and $b = 1$.

(i) $H_1(s) = \dfrac{s+1}{s+1}$




(ii) $H_2(s) = \dfrac{s-1}{s+1}$




(c) The magnitude of the frequency response function is not affected by whether a zero is located in the left-half plane or its relection about the imaginary axis. The phase response is significantly affect however. In general the phase-shift associated with a right-half plane zero is greater than that of the corresponding left-half plane position - this can be easily demoinstrated using the geometric interpretation from the pole-zero plot. Hence the name "non-minimum phase" system.

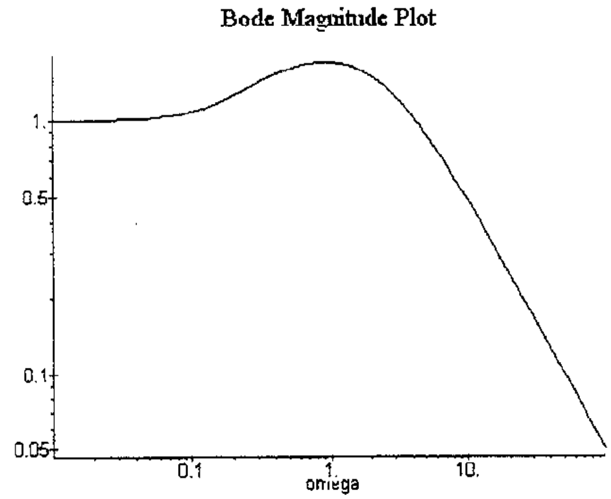
(d) If

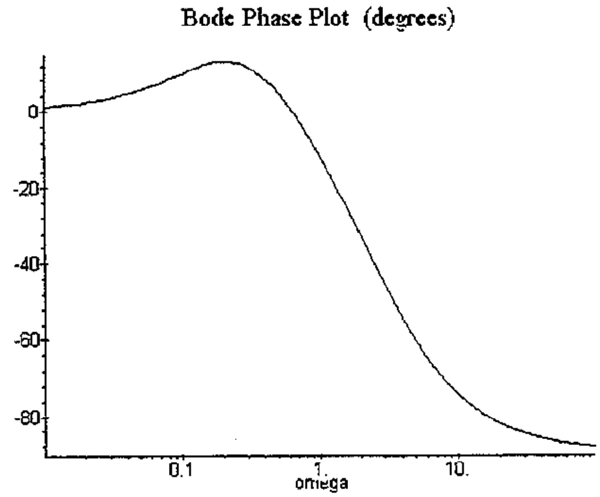
$$H_2(s) = \frac{s-a}{s+a}$$

then

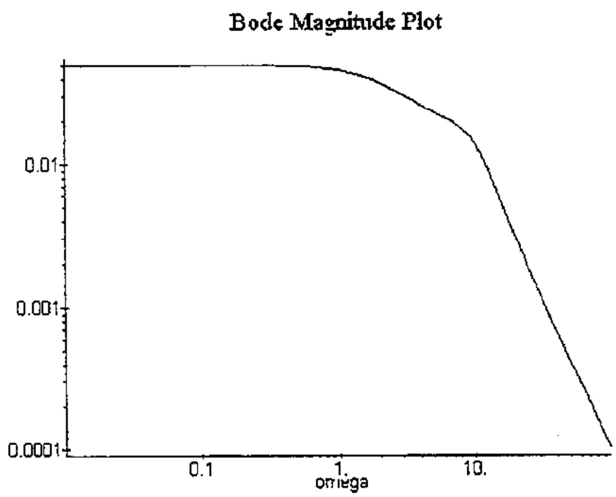
$$|H_2(j\omega)| = \frac{\sqrt{(j\omega)^2 + a^2}}{\sqrt{(j\omega)^2 + a^2}} = 1 \quad \text{and} \quad \angle H(j\omega) = \pi - 2tan^{-1}(j\omega/a)$$

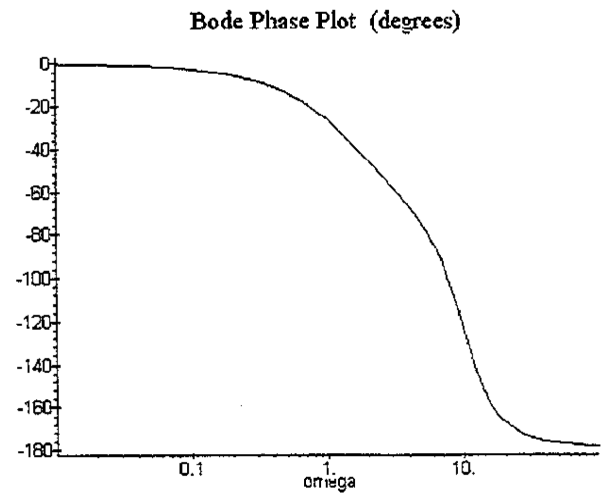
(c) $H_3(s) = \dfrac{5s+1}{s^2+3s+1}$




(d) $H_4(s) = \dfrac{s+10}{(s+2)(s^2+10s+100)}$




## Problem 14.9

The solutions provided are based on the geometric interpretation of the frequency response (Sec 14.7).

(a) The break points are determined by the radial distance of the poles and zeros from the origin. The highest frequency break frequency is therefore determined by the most distant pole or zero: (a) 5 rad/s, (b) 156.2 rad/s, (c) 3.162 rad/s, (d) 12 rad/s. The slope of the high frequency asymptote is $-(n_p - n_z) * 20$ dB/decad where $n_p$is the number of system poles and $n_z$ is the number of system zeros: (a) -20dB/decade, (b) -40dB/decade, (c) -40 dB/decade, (d) -20dB/decade.

**(b)** The asymptotic high frequency phase response is $(n_z - n_p) * \pi/2$ rad: (a) $-\pi/2$ rad, (b) $-\pi$ rad, $-\pi$ rad, $-\pi/2$ rad.

**(c)** The low frequency asymptotic behavior is determined by poles or zeros at the origin: (a) the low frequency response tends to a constant value, (b) the low frequency response tends to infinity, (c) the low frequency response tends to zero, (d) while in principle the low frequency response tends to a constant value, this is a marginally stable system.

**(d)** The low frequency phase shift is determined by the contribution from each pole and zero : left-half plane poles/zeros do not contribute, right half plane zeros contribute $\pi$ rad, zeros and poles at the origin contribute $\pm\pi/2$ rad. (a) 0 rad, (b) $-\pi/2$ rad, (c) $+3\pi/2$ rad, (d) 0 rad.

## Problem 14.10

Note: There is an error in early printings of the book – the pole locations should read:

> ...The $N$th-order Butterworth filter has its $N$ poles equally spaced on a circle in the left-half $s$-plane,
>
> $$p_n = \omega_c e^{j\pi(2n+N-1)/2N}, \qquad n = 1, 2, \ldots, N$$
>
> where $p_n$ is the $n$th pole location, $\omega_c$ is the half-power (-3dB) cut-off frequency of the filter...

We continue with the problem as intended, if the incorrect form is assumed the effect is to redefine attenuation at the filter cut-off frequency $\omega_c$.

**(a)** The poles are located at

$$\begin{aligned} p_1 &= \omega_c e^{j3\pi/4} = -\frac{\sqrt{2}}{2}\omega_c + j\frac{\sqrt{2}}{2}\omega_c = -0.707\omega_c + j0.707\omega_c \\ p_2 &= \omega_c e^{j5\pi/4} = -\frac{\sqrt{2}}{2}\omega_c - j\frac{\sqrt{2})}{2}\omega_c = -0.707\omega_c - j0.707\omega_c \end{aligned}$$

The unity gain second order system is

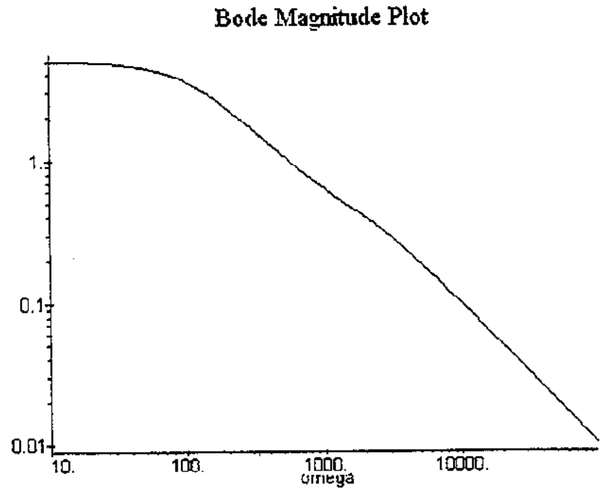
$$H(s) = \frac{p_1 p_2}{(s-p_1)(s-p_2)} = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

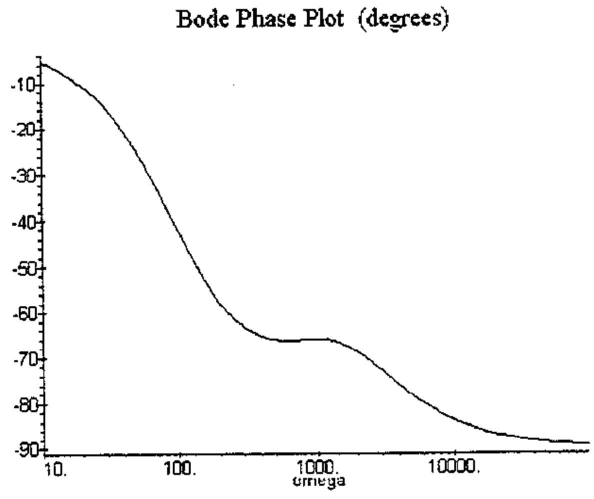
the damping ratio $\zeta = 0.707$, and the undamped natural frequency $\omega_n = \omega_c$ rad/s.

**(b)** For the third-order filter ($N = 3$) with $\omega_c = 2\pi \times 100$ rad/s, the poles will be at locations

$$\begin{aligned} p_1 &= (\omega_c)\, e^{j4\pi/6} = -314.2 + j544.1 \\ p_2 &= (\omega_c)\, e^{j6\pi/6} = -628.3 \\ p_3 &= (\omega_c)\, e^{j8\pi/6} = -314.2 - j544.1 \end{aligned}$$

**(b)** The Bode plots are:




**(c)**

$$H(j\omega) = \frac{1000(j\omega + 1000)}{(j\omega + 2000)(j\omega + 100)}$$

At low frequencies $|j\omega| \to 5$ while at high frequencies $|j\omega| \to 1000/\omega$.

## Problem 14.13

**(a)** At the -3 dB frequency

$$|H(j\omega)| = \frac{K}{\sqrt{(\omega\tau)^2 + 1}} = \frac{\sqrt{2}K}{2}$$

which is solved to give $\omega = 1/\tau$. If $\tau = 1$, then $\omega = 1$.

**(b)** The overall transfer function is

$$H_1(s) = K_a \frac{\tau_1 s + 1}{\tau_2 s + 1} \frac{K}{\tau s + 1}$$

If $\tau_1$ is adjusted to be 1 s, there is pole-zero cancellation and the transfer function becomes

$$H_1(s) = \frac{KK_a}{\tau_2 s + 1}$$

and the cut-off frequency of the system is $1/\tau_2$. To double the cut-off frequency from 1 rad/s to 2rad/s, $\tau_2 = 0.5$. The -20 dB frequency $\omega_{0.1}$ is found from

$$|H(j\omega_{0.1})| = \frac{KK_a}{\sqrt{(0.5\omega_{0.1})^2 + 1}} = 0.1KK_a$$

or $\omega_{0.1} = \sqrt{396} = 19.9$ rad/s, and the response is attenuated by more than 20 dB for frequencies greater than 19.1 rad/s.

**Problem 14.14**

(a) The transfer functions $H_1(s)$ relating the force in the spring $F_K$ to the input force $F$, and $H_2(s)$ relating the damper force $F_B$ to the input force are

$$H_1(s) = \frac{K}{ms^2 + Bs + K}, \qquad H_2(s) = \frac{Bs}{ms^2 + Bs + K}$$

The force transmitted to the care frame is $F = F_K + F_B$, so that the overall transfer function is

$$H(s) = H_1(s) + H_2(s) = \frac{(B/m)s + K/m}{s^2 + (B/m)s + K/m}$$

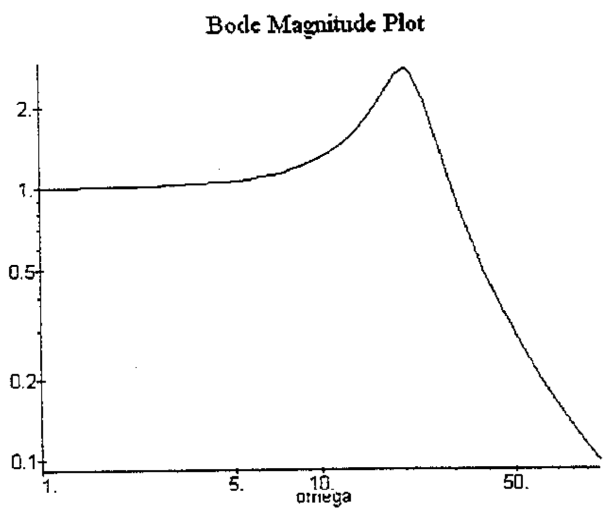
(b) For a second-order system the resonant frequency is (p.464) $\omega_{peak} = \omega_n\sqrt{(1 - 2\zeta^2}$, and for the given $\zeta = 0.2$, $\omega_{peak} = 0.959\omega_n$. Then $\omega_n = \sqrt{K/m} = 1.043\omega_{peak}$, giving

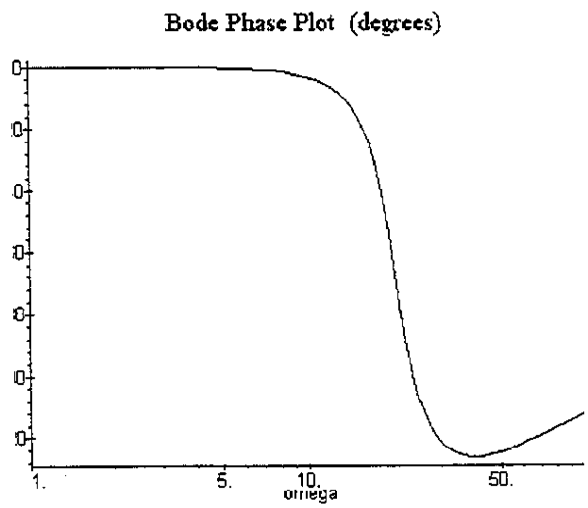
$$K = (1.043\omega_{peak})^2 m = 435,200 \text{ N/m}$$

(c) The value of $B$ must first be found. From part (a) $\omega_n = 1.043\omega_{peak} = 20.86$ rad/s. Also $2\zeta\omega_n = B/m$ with $\zeta = 0.2$ gives $B = 8728$ N-s/m. The overall transfer function is

$$H(s) = \frac{8.344s + 435.2}{s^2 + 8.344s + 435.2}$$

The Bode plots for this system are shown below.

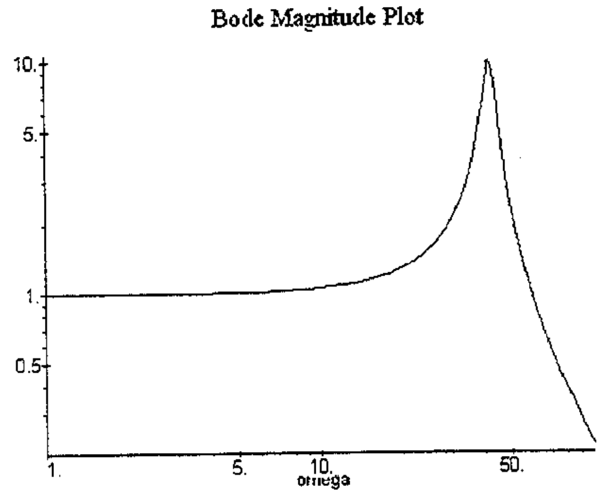


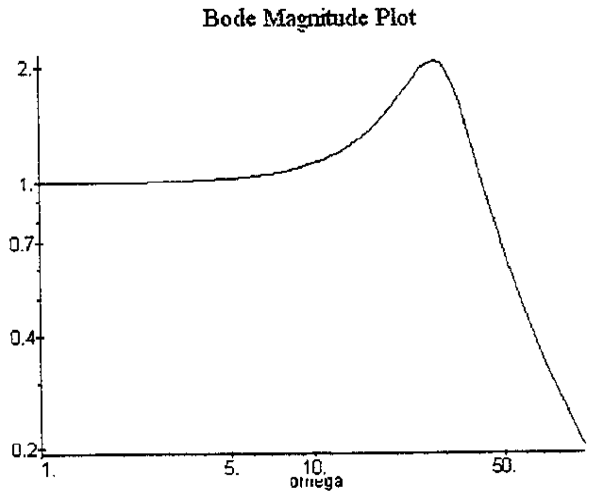




(c) The specification for the alternative mounts, together with the computed damping ratios and natural frequencies are summarized in the following table:

| | | | | |
|---|---|---|---|---|
| Mount A: (original) | $K_A = K$ | $B_A = B$ | $\zeta = 0.2$ | $\omega_n = 20.86$ |
| Mount B: | $K_B = 4K,$ | $B_B = 0.5B$ | $\zeta = 0.05$ | $\omega_n = 41.72$ |
| Mount C: | $K_C = 2K,$ | $B_C = 2B$ | $\zeta = 0.283$ | $\omega_n = 29.50$ |

Bode magnitude plots of the transmissibility force ratio $H(j\omega)$ for mounts B and C are shown below.




The question of whether mounts B or C offer improvement depends on the nature of the exciting forces. In general C provides higher damping and a higher resonant frequency and may offer better performance in practice. Mount B, with its low damping shows a very pronounced resonance at around 40 rad/s.

### Problem 14.15

From Ch. 13 The impedance of an inductance is $Z_L(s) = sL$, and the impedance of a capacitance is $Z_C(s) = 1/sC$. For the parallel LC circuit:

$$Z_p(s) = \frac{Z_L(s)Z_C(s)}{Z_L(s)+Z_C(s)} = \frac{(sL)/(sC)}{sL+(1/sC)} = \frac{Ls}{LCs^2+1}$$

and for the series connection

$$Z_s(s) = Z_L(s) + Z_C(s) = sL + 1/(sC) = \frac{LCs^2+1}{s}$$

The sinusoidal input impedance $Z(j\omega) = Z(s)|_{s=j\omega}$

$$Z_p(j\omega) = \frac{jLC\omega}{1-LC\omega^2}, \qquad Z_s(j\omega) = \frac{-j\,(1-LC\omega^2)}{\omega}$$

(a) By inspection

(i) as $\omega \to 0$,

$$\begin{aligned} Z_p(j\omega) &\to 0 \\ Z_s(j\omega) &\to \infty \end{aligned}$$

((ii) as $\omega \to \infty$,

$$\begin{aligned} Z_p(j\omega) &\to 0 \\ Z_s(j\omega) &\to \infty \end{aligned}$$

((iii) when $\omega = \sqrt{1/LC}$,

$$\begin{aligned} Z_p(j\omega) &= \infty \\ Z_s(j\omega) &= 0 \end{aligned}$$

(b) At a frequency $\omega = \sqrt{L/C}$ the impedances are $Z_c(j\omega) = -j\sqrt{L/C}$ and $Z_s(j\omega) = j\sqrt{L/C}$, that is they are conjugates.

In the parallel connection there is a common voltage across the inductance and capacitance. Using Kirchoff's current law at either node net current flow into the circuit is

$$i(t) = i_L(t) + i_C(t) = \left[\frac{1}{Z_L(s)} + \frac{1}{Z_C(s)}\right]\sin(\omega t) = 0$$

or in other words, no net current flows into the circuit even though significant current is flowing in the two elements and a finite voltage appears at the terminals.

In the series connection the currents through the inductor and capacitor are the same, but the the terminal voltage is

$$V(t) = v_l(t) + v_C(t) = (Z_L(j\omega) + Z_C(j\omega))\, I_0 \sin(\omega t) = 0$$

that is while $v_C(t)$ and $v_L(t)$ are both finite, they sum together to zero at the input.

When the coil has finite resistance $R$ (in series) as well as inductance the series connection has impedance

$$Z_s(s) = Z_L(s) + Z_C(s) + Z_R(s) = sL + 1/(sC) + R = \frac{LCs^2 + RCs + 1}{Cs}$$

and the sinusoidal impedance is

$$Z_s(j\omega) = \frac{-j}{LC}\frac{(1 - LC\omega^2) + jRC\omega}{C\omega}$$

Then (i) as $\omega \to 0 \quad Z_s(j\omega) \to \infty$, (ii) as $\omega \to \infty \quad Z_s(j\omega) \to \infty$, and (iii) when $\omega = \sqrt{(1/LC)}$, $Z_s(j\omega) = R$.

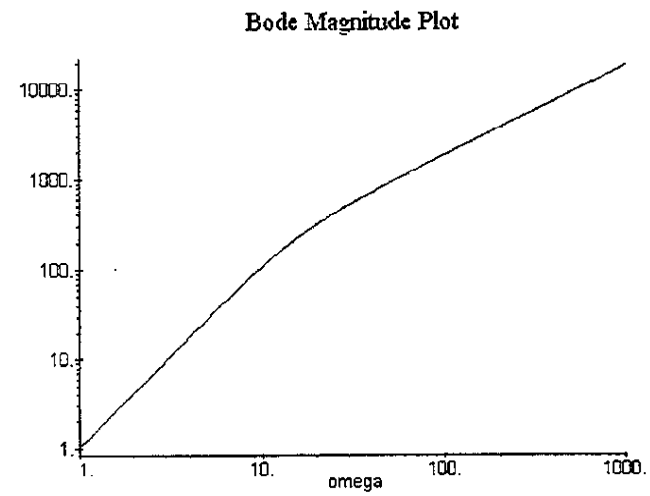
For the parallel connection

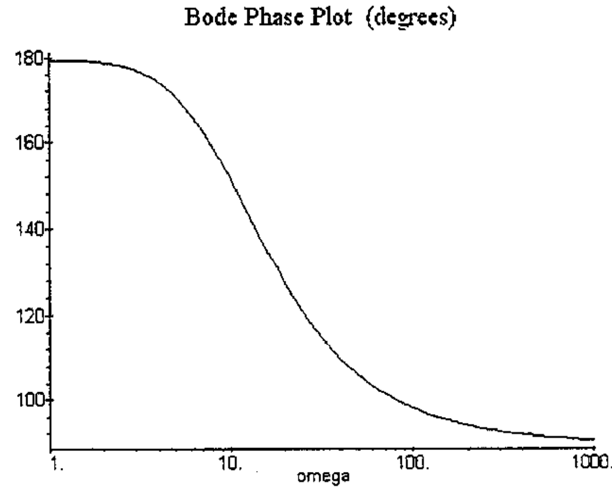
$$Z_p(s) = \frac{(R + sL)/sC}{sL + R + 1/sC} = \frac{R + sL}{LCs^2 + RCs + 1}$$

and

$$Z_p(j\omega) = \frac{R + j\omega L}{(1 - LC\omega^2) + jRC\omega}$$

Then (i) as $\omega \to 0 \ \ Z_p(j\omega) \to R$, (ii) as $\omega \to \infty \ \ Z_s(j\omega) \to 0$, and (iii) when $\omega = \sqrt{(1/LC)}$, $|Z_s(j\omega)| = (R^2 + L/C)/(R\sqrt{C/L})$.

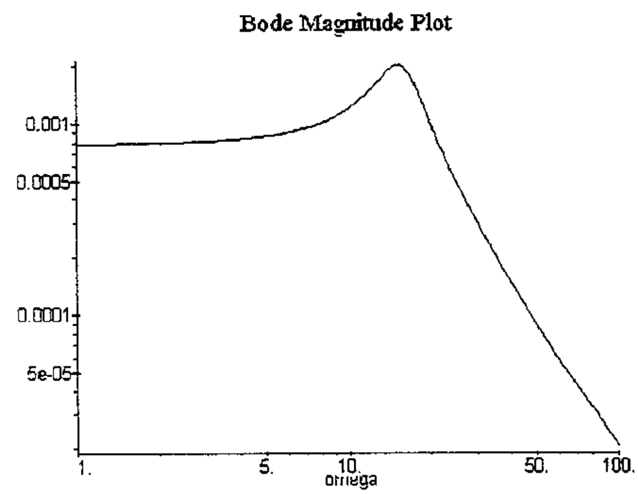


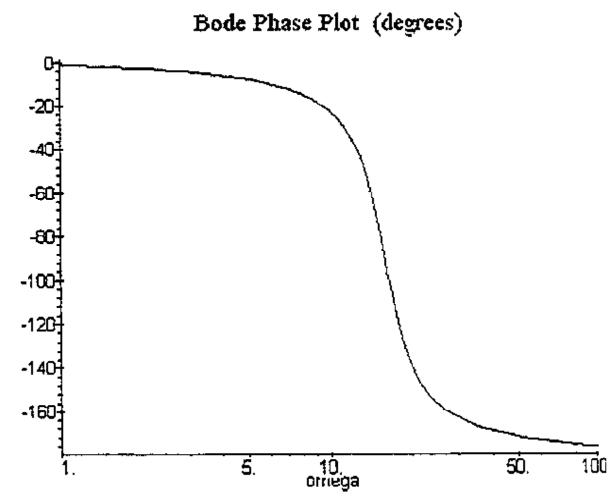




## Problem 14.21

(a) The following Bode plots are derived from the given transfer function

$$H(s) = \frac{0.2}{s^2 + 6.4s + 256}$$



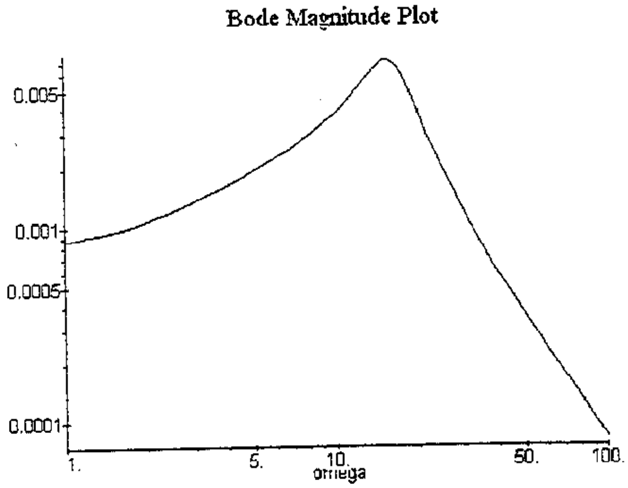



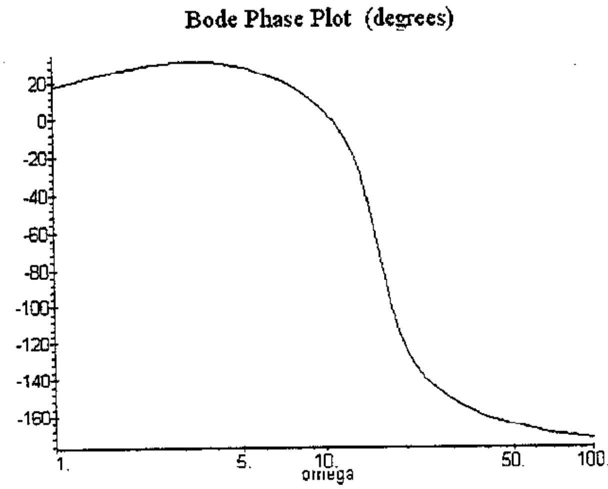

Any noise components with frequencies in the range of approximately 9 – 30 rad/s will be amplified by the system.

(b) With the filter with the transfer function

$$H_f(s) = \frac{0.5s + 1}{0.125s + 1}$$

the Bode plots are:

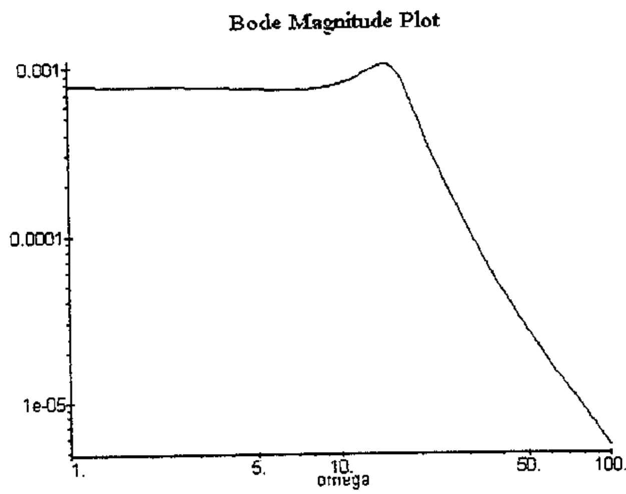


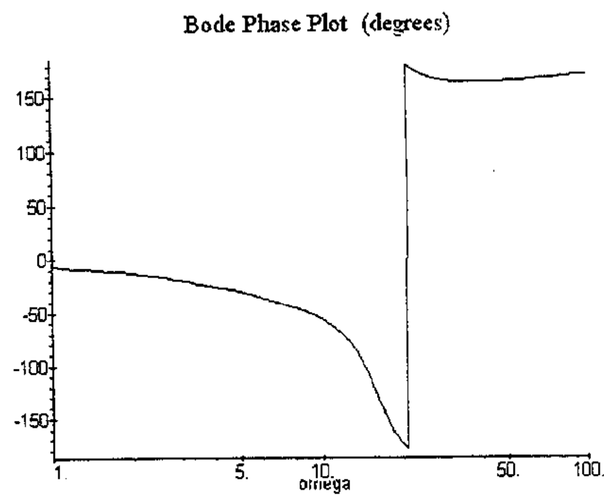




and with

$$H_f(s) = \frac{0.03125s + 1}{0.125s + 1}$$

the Bode plots are:




Clearly the last case reduces the effects of the system resonance in the overall response.

## Problem 14.22

(a) The transfer function between the pump output flow $Q(t)$ and the pressure at the pump outlet is:

$$H(s) = (sI + R)$$

(b) The system frequency response is

$$H(j\omega) = j\omega I + R, \quad |H(j\omega)| = \sqrt{(I\omega)^2 + R^2}, \quad \angle H(j\omega) = \tan^{-1}\frac{\omega I}{R}$$