The Transfer Function

Problem 12.1

(a) $$H(s) = \frac{s+2}{s+3}$$

Pole: $s = -3$
Zero: $s = -2$

(b) $$H(s) = \frac{2s+1}{s^2+7s+12} = 2\frac{s+0.5}{(s+3)(s+4)}$$

Poles: $s = -3 \qquad s = -4$
Zero: $s = -0.5$

(c) $$H(s) = \frac{s}{s^2+5s+7} = \frac{s}{(s+(5/2+j\sqrt{3}/2))(s+(5/2-j\sqrt{3}/2))}$$

Poles: $s = -5/2 + j\sqrt{3}/2 \quad s = -5/2 - j\sqrt{3}/2$
Zero: $s = 0$

Problem 12.4

(a) The system transfer function relating $y(t)$ to $u(t)$ is

$$H(s) = (H_1(s) + H_2(S))\, H_3(s)$$

(b) Assume the transfer functions $H_1(s)$ and $H_2(s)$ are:

$$H(s) = \left(\frac{2}{s+1} + \frac{s}{s+2}\right)\frac{1}{s+3} = \frac{s^2+3s+4}{s^3+6s^2+11s+6}$$

System poles: s$=-1, -2, -3$
System zeros: s $= -3/2 \pm j\sqrt{7}/2$

The differential equation is

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = \frac{d^2u}{dt^2} + 3\frac{du}{dt} + 4u$$

Problem 12.12

$$H_1(s) = \frac{s^2+16}{s^3+12s^2+41s+30} \qquad H_2(s) = \frac{s^2+4s+25}{s^3+12s^2+41s+30}$$

(a) For the first system the zeros are $z_1, z_2 = \pm j4$. The input

$$\sin(4t) = \frac{1}{2j}\left(e^{j4t} - e^{-j4t}\right)$$

For a complex exponential input $e^{j\omega t}$ the particular response is $y_p(t) = H(j\omega)e^{j\omega t}$ and for the sinusoid

$$y_p(t) = \frac{1}{2j}\left(H(j4)e^{j4t} - H(-j4)e^{-j4t}\right) = 0$$

since $H(j4) = H(-j4) = 0$.

(b) For the second system the zeros are $z_1, z_2 = -2 \pm j5$. The input

$$e^{-2t}\sin(5t) = \frac{1}{2j}\left(e^{(-2+j5)t} - e^{(-2-j5)t}\right)$$

For a complex exponential input e^{jst} the particular response is $y_p(t) = H(s)e^{st}$ and for the sinusoid

$$y_p(t) = \frac{1}{2j}\left(H(-2+j5)e^{(-2+j5)t} - H(-2-j5)e^{(-2-j5)t}\right) = 0$$

since $H(-2+j5) = H(-2-j5) = 0$.

(c) The particular response of any system with a pair of complex conjugate zeros at $z_1, z_2 = -\sigma \pm j\omega$ to an input $u(t) = e^{(-\sigma \pm j\omega)t}$ will be zero because $H(-\sigma \pm j\omega) = 0$.

Problem 12.15

The pole-zero plots are not given here.

(a) Pole as $s = -4$, zero at $s = 0$. The system is stable. The initial condition response will be of the form

$$y(t) = Ce^{-4t}$$

(b) Poles at $s = -1$ and $s = -4$. The system is stable. The initial condition response is of the form

$$y(t) = C_1 e^{-t} + C_2 e^{-4t}$$

(c) Poles at $s = -2$ and $s = +2$. The system is unstable. The initial condition response is of the form

$$y(t) = C_1 e^{-2t} + C_2 e^{2t}$$

(d) Poles at $s = -1 + j\sqrt{(3)}$ and $s = -1 - j\sqrt{(3)}$, and a zero at $s = +2$. This is an underdamped non-minimum phase system – it is stable. The initial condition response is of the form

$$y(t) = Ce^{-t}\sin(3t + \phi$$

where C and ϕ are found from the initial conditions.

Problem 12.22

(a) The characteristic equation is $s^2 + 2s + 1 = 0$. The poles are $s_1, s_2 = -1, -1$, each with a time constant of 1 sec. Because of the repeated root there is only one modal component e^{-t}, but it appears in the homogeneous response as

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}$$

(b) The characteristic equation is $s^2 + s + 4 = 0$. The poles are $s_1, s_2 = -0.5 \pm 0.5\sqrt{15}$. The equivalent second order parameters are $\omega_n = 2$, $\zeta = 0.25$. The homogeneous response is of the form

$$y_h(t) = Ae^{-0.5t} \sin(1.936t + \phi)$$

where A and ϕ are to be determined from the initial conditions.

(c) The characteristic equation is $s^2 - s + 4 = 0$. The poles are $s_1, s_2 = 0.5 \pm 0.5\sqrt{15}$. The system is unstable, but equivalent second order parameters are $\omega_n = 2$, $\zeta = -0.25$. The homogeneous response is of the form

$$y_h(t) = Ae^{0.5t} \sin(1.936t + \phi)$$

where A and ϕ are to be determined from the initial conditions.

(d) The characteristic equation is $\det[s\mathbf{I} - \mathbf{A}] = s^3 - s^2 + 2s - 2 = 0$. The poles are $s_1, s_2 = \pm j\sqrt{2}$ and $s_3 = 1$. The real (unstable) pole has a time constant of 1 sec. The equivalent second order parameters are $\omega_n = 1.414$, $\zeta = 0$. The homogeneous response is of the form

$$y_h(t) = Ae^{t} + B \sin(1.414t + \phi)$$

where A , B, and ϕ are to be determined from the initial conditions. The system is unstable.

(e) The characteristic equation is $\det[s\mathbf{I} - \mathbf{A}] = s^3 + s^2 + 2s + 2 = 0$. The poles are $s_1, s_2 = \pm j\sqrt{42}$ and $s_3 = -1$. The real pole has a time constant of 1 sec. The equivalent second order parameters are $\omega_n = 1.414$, $\zeta = 0$. The homogeneous response is of the form

$$y_h(t) = Ae^{-t} + B \sin(1.414t + \phi)$$

where A , B, and ϕ are to be determined from the initial conditions. The system is stable.

Problem 12.24

(a) The transfer function is

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B} = \frac{\mathbf{C}\mathrm{adj}\left[s\mathbf{I}-\mathbf{A}\right]\mathbf{B}}{\det\left[s\mathbf{I}-\mathbf{A}\right]}$$

$$= \frac{\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix}}{s^2+6s+8}$$

$$= \frac{8}{s^2+6s+8}$$

(b) Using the results of Prob. 12.5,

$$H_{cl}(s) = = \frac{H_1(s)}{1+H_1(s)H_2(s)}$$

$$= \frac{8}{s^2+6s+8(1+K)}$$

(c) The characteristic equation is

$$s^2+6s+8(1+K)=0$$

and its roots are

$$s_1, s_2 = -3 \pm \sqrt{1-8K}$$

The poles are real and negative for $0 < K < 0.125$. For $K > 0.125$ they are complex conjugates, $s_1, s_2 = -3 \pm j\sqrt{8K-1}$. The locus of the roots is shown below