# Home work # 1

$\boxed{10.8\ (c)}$

$$A = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix} \rightarrow [A - \lambda I] = \begin{bmatrix} -1-\lambda & -1 \\ 4 & -1-\lambda \end{bmatrix}$$

$$\det[A - \lambda I] = 1 + 2\lambda + \lambda^2 + 4 = 0 \rightarrow \lambda^2 + 2\lambda + 5 = 0$$

characteristic eq.

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i \qquad \text{Eigen Values}$$

To find the damping ratio and natural freq. compare your characteristic eq with $\lambda^2 + 2\xi_n \omega_n \lambda + \omega_n^2 = 0$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\left.\begin{aligned} 2\xi_n \omega_n = 2 &\rightarrow \xi_n \omega_n = 1 \\ \omega_n^2 = 5 &\rightarrow \omega_n = \sqrt{5} \end{aligned}\right\} \rightarrow \begin{aligned} \omega_n &\simeq 2.24 \\ \xi_n &\simeq 0.447 \end{aligned}$$

$\boxed{10.12\ (c)}$

$$A = \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix} \rightarrow [A - \lambda I] = \begin{bmatrix} -\lambda & 1 \\ -25 & -\lambda \end{bmatrix}$$

$\det[A - \lambda I] = 0 \longrightarrow \lambda^2 + 25 = 0$ characteristic eq.

$\longrightarrow \lambda = \pm 5i$ Eigen values

$$\lambda_1 = 5i : [A - 5i\,I]\,v_1 = 0 \rightarrow \begin{bmatrix} -5i & 1 \\ -25 & -5i \end{bmatrix} v_1 = 0 \longrightarrow v_1 = \begin{bmatrix} 1 \\ -5i \end{bmatrix}$$

$$\lambda_2 = -5i : [A + 5i\,I]\,v_2 = 0 \rightarrow \begin{bmatrix} 5i & 1 \\ -25 & 5i \end{bmatrix} v_2 = 0 \rightarrow v_2 = \begin{bmatrix} 1 \\ 5i \end{bmatrix}$$

Eigen Vectors

From here we get the modal matrix : $M = [\, v_1 \,\vdots\, v_2 \,]$

$$M = \begin{bmatrix} 1 & 1 \\ -5i & 5i \end{bmatrix} \qquad e^{\Lambda t} = \begin{bmatrix} e^{-5i} & 0 \\ 0 & e^{5i} \end{bmatrix}$$

$\boxed{10.14}$ $\dot{x} = Ax + Bu$ $\qquad y = Cx + Du$

$$\left.\begin{aligned} \dot{x}_1 &= -4x_1 \\ \dot{x}_2 &= -x_2 \\ \dot{x}_3 &= -7x_3 \end{aligned}\right\} \rightarrow A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$y = 2x_1 + 3x_2 - x_3 \rightarrow C = [\, 2 \quad 3 \quad -1 \,]$$

The system is diagonal so we can directly find the state transition matrix.

$$\phi(t) = e^{\Lambda t} = \begin{bmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-7t} \end{bmatrix}$$

and also given the initial conditions the output is:

$$y(t) = C\,\phi(t)\,x(0) = 10\,e^{-4t} + 45\,e^{-t} + (-2)\,e^{-7t}$$

$\boxed{10.15}$

The system is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$[A - \lambda I] = \begin{bmatrix} -2-\lambda & 2 \\ 2 & -3-\lambda \end{bmatrix} \rightarrow \det[A - \lambda I] = 0 \rightarrow (-2-\lambda)(-3-\lambda) - 2\times 2 =$$

$$\rightarrow 6 + 5\lambda + \lambda^2 - 4 = 0 \longrightarrow \lambda^2 + 5\lambda + 2 = 0 \rightarrow \lambda_1 = -0.438 \quad \text{Eigen values}$$

$$\lambda_2 = -4.562$$

From here using eigen values we can find eigen vectors

$$\lambda_1 = -0.438 \longrightarrow \begin{bmatrix} -2+0.438 & 2 \\ 2 & -3+0.438 \end{bmatrix} \underbrace{\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}}_{M_I} = 0 \rightarrow M_I = \begin{bmatrix} 1 \\ 0.781 \end{bmatrix}$$

$$\lambda_2 = -4.562 \rightarrow \begin{bmatrix} -2+4.562 & 2 \\ 2 & -3+4.562 \end{bmatrix} \underbrace{\begin{bmatrix} m_1' \\ m_2' \end{bmatrix}}_{M_{II}} = 0 \rightarrow M_{II} = \begin{bmatrix} 1 \\ -1.28077 \end{bmatrix}$$

Modal matrix: $M = [M_I \mid M_{II}]$

$$M = \begin{bmatrix} 1 & 1 \\ 0.781 & -1.281 \end{bmatrix} \qquad e^{\Lambda} = \begin{bmatrix} e^{-0.438t} & 0 \\ 0 & e^{-4.562} \end{bmatrix}$$

$$M^{-1} = \frac{1}{\det[M]} \times \begin{bmatrix} & - \\ - & \end{bmatrix} = \begin{bmatrix} 0.621 & 0.485 \\ 0.379 & -0.485 \end{bmatrix}$$

The diagonal form is found from Eq. (10.71)

$$\dot{x}' = M^{-1} A M x' + M^{-1} B u$$

$$\dot{x}' = \begin{bmatrix} -0.438 & 0 \\ 0 & -4.562 \end{bmatrix} x' + \begin{bmatrix} 0.485 \\ -0.485 \end{bmatrix} u(t)$$

We also have to apply new initial conditions:

$$x'(0) = M^{-1} x(0)$$

$\boxed{10.18}$

From Eq. (10.86) the general step response is:

$$y(t) = C A^{-1} \left[ e^{At} - I \right] B k + D k u(t)$$

for this special case $a = -3$, $b = 2$, $c = 2$, $d = 1$, $k = 1$

$$y_{step}(t) = \frac{7}{3} - \frac{4}{3} e^{-3t}$$

The impulse response is given: Eq (10.81)

$$y(t) = C e^{At} \left[ x(0) + B k \right] + D k \delta(t)$$

The same way we get $y_{impulse}(t) = 4 e^{-3t} + \delta(t)$

The ramp response is given: Eq (10.92)

$$y(t) = C e^{At} x(0) + C A^{-1} \left[ A^{-1} (e^{At} - I) - I t \right] B k + D k t$$

so we get: $y_{ramp}(t) = \frac{4}{9} e^{-3t} - \frac{4}{9} + \frac{7}{3} t$