Stability of systems:

The solution of $\dot{\underset{\sim}{x}} = A \underset{\sim}{x}$ is given by

$$\underset{\sim}{x} = e^{At} x_0$$

For stability we want the solution to decay to zero (or at least not grow to ∞) as $t \to \infty$

Since

$$\underset{\sim}{x} = e^{At} \underset{\sim}{x}_0 = M \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix} M^{-1} \underset{\sim}{x}_0$$

it is clear that <u>for stability</u> we must require

$$\boxed{Re(\lambda_i) \leq 0}$$ for <u>all</u> i.

Response of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + B\underset{\sim}{u}$ to various input functions

Impulse:

$$\underset{\sim}{u}(t) = K \cdot \delta(t) = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \cdot \delta(t)$$ delta function at $t=0$.

$$\Rightarrow \quad \underset{\sim}{x}(t) = e^{At} \cdot x(0) + \int_0^t e^{A(t-\tau)} \cdot B \cdot K \cdot \delta(t)\, d\tau$$

But we know that

$$\int_a^b g(\tau) \cdot \delta(\tau - c)\, d\tau = g(c) \text{ where } \quad a < c < b$$

so

$$\underset{\sim}{x}(t) = e^{At} \cdot x(0) + e^{At} \cdot B \cdot K$$

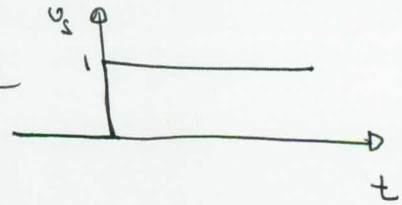
$$\boxed{x(t) = e^{At}[x(0) + BK]}$$ ← For impulse at $t=0$

also for impulse at $t = t_0$

$$\boxed{x(t) = e^{At}[x(0) + e^{-t_0} \cdot B \cdot K]}$$ for $t > t_0$

Step:

$$U(t) = K\cdot u_s(t) = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}\cdot u_s(t)$$ where



$\Rightarrow$

$$x = e^{At}\cdot x_0 + \int_0^t e^{A(t-\tau)} B\cdot K u_s(\tau)\,d\tau$$

but $u_s = 1$ for $t > 0$ $\Rightarrow$

$$x = e^{At}\cdot x_0 + \int_0^t e^{A(t-\tau)}\cdot B\cdot K\,d\tau$$

$$x = e^{At}\cdot x_0 + \int_0^t e^{A(t-\tau)}\cdot d\tau\; B\cdot K$$

$$x = e^{At}\cdot \cancel{x_0} + A^{-1}e^{A(t-\tau)}\Big|_0^t BK = e^{At}x_0 + A^{-1}(I - e^{At})\cdot B\cdot K$$

$$\boxed{x = e^{At}\cdot x_0 + A^{-1}(e^{At} - I)BK}$$ $\longleftarrow$ for step

Note:

$$\int e^{A\tau}d\tau = \int M\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t}\end{bmatrix}M^{-1}d\tau = M\begin{bmatrix} \frac{1}{\lambda_1}e^{\lambda_1 t} & & & 0 \\ & \frac{1}{\lambda_2}e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\lambda_n}e^{\lambda_n t}\end{bmatrix}M^{-1}$$

$$= M\begin{bmatrix} \frac{1}{\lambda_1} & & & 0 \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\lambda_n}\end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & 0 \\ 0 & & \ddots & \\ & & & e^{\lambda_n t}\end{bmatrix}M^{-1} = M\begin{bmatrix} \frac{1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_n}\end{bmatrix}M^{-1}M\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t}\end{bmatrix}M^{-1}$$

$$= \left[M\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}M^{-1}\right]^{-1}\cdot M\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t}\end{bmatrix}M^{-1} = A^{-1}\cdot e^{A\tau}$$

Ramp

$$u(t) = K\cdot t = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}\cdot t$$

then

$$\boxed{x = e^{At}\,\underset{\sim}{x}(0) + A^{-1}\left[A^{-1}\left(e^{At}-I\right) - It\right]BK}$$

This comes from:

$$x = e^{At}\cdot \underset{\sim}{x}_0 + \int_0^t e^{A(t-\tau)}\cdot BK\tau\cdot d\tau$$

$$x = e^{At}\cdot \underset{\sim}{x}_0 + \int_0^t \tau\, e^{A(t-\tau)}\,d\tau\cdot B\cdot K$$ integrate by parts!

$$x = e^{At}\cdot \underset{\sim}{x}_0 + \left[\tau(-A^{-1})e^{A(t-\tau)}\Big|_0^t - \int_0^t e^{A(t-\tau)}\,d\tau\right]B\cdot K$$

$$x = e^{At}\cdot \underset{\sim}{x}_0 + \left[-t A^{-1} - 0(-A^{-1})e^{At} - A^{-1}e^{A(t-\tau)}\Big|_0^t\right]BK$$

$$x = e^{At}\cdot \underset{\sim}{x}_0 + A^{-1}\left[-t\,I - A^{-1}e^{A\cdot 0} + A^{-1}e^{At}\right]B\cdot K$$

$$\boxed{x = e^{At}\cdot \underset{\sim}{x}_0 + A^{-1}\left[A^{-1}\left(e^{At}-I\right) - It\right]\cdot B\cdot K}$$ ✓