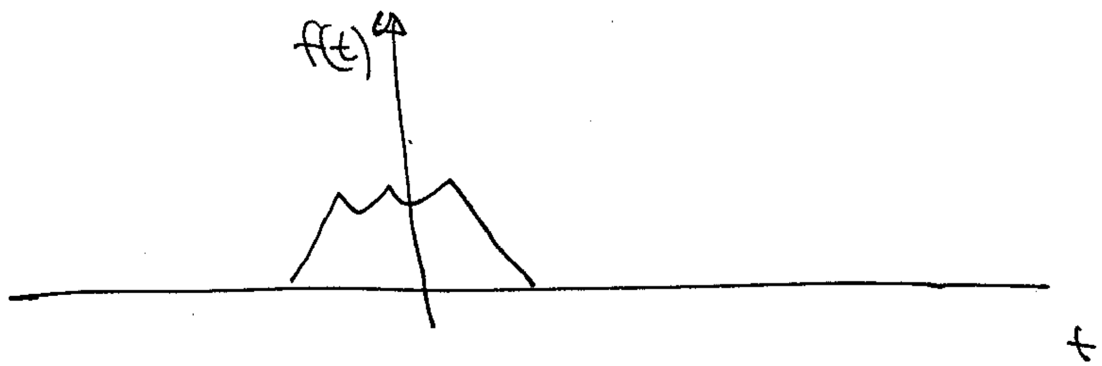
Fourier Transform

We need a way to handle non periodic inputs to systems.

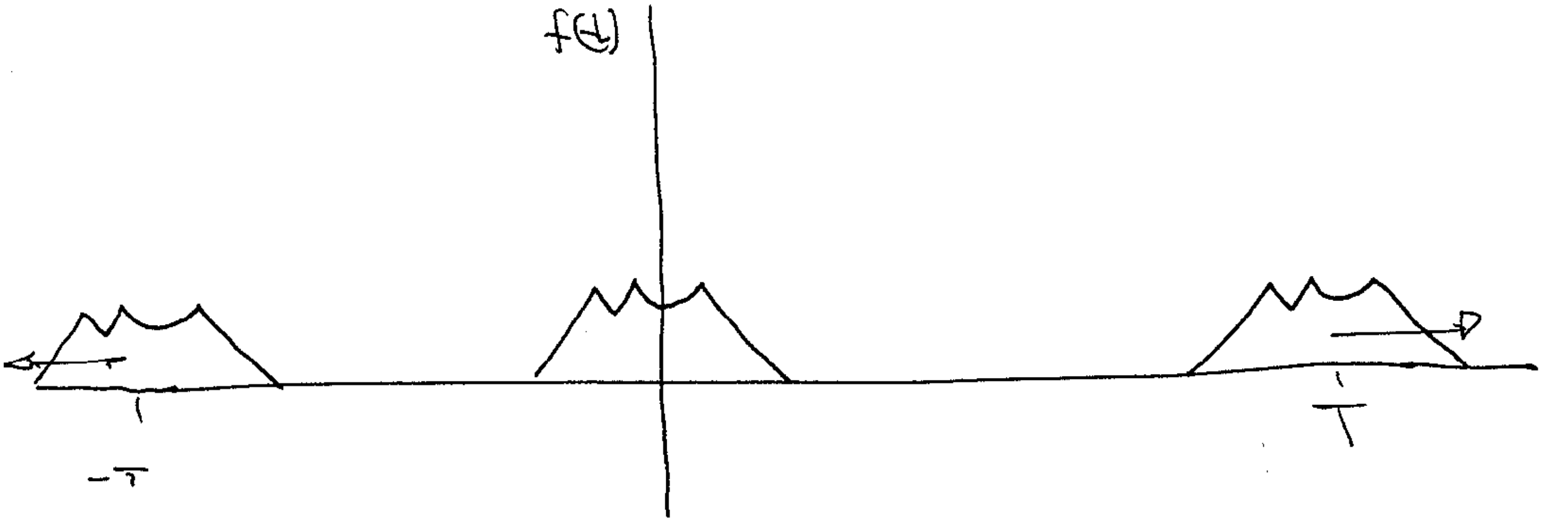
Fourier series offer a powerful method to handle periodic function. But what if the input is complex and non periodic?

Methodology:

Assume we have



then we make this function periodic



We let $T \to \infty$

so from Fourier series we have:

$$\left.\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} F_n \cdot e^{jn\omega_0 t} \\ F_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\cdot e^{-jn\omega_0 t}\,dt \end{aligned}\right\} \text{Fourier } \underline{\text{series}} \text{ pair}$$

as $T \to \infty$ $\quad \omega_0 = \frac{2\pi}{T} \to d\omega \quad$ and $\quad n\omega_0 \to \omega$

so

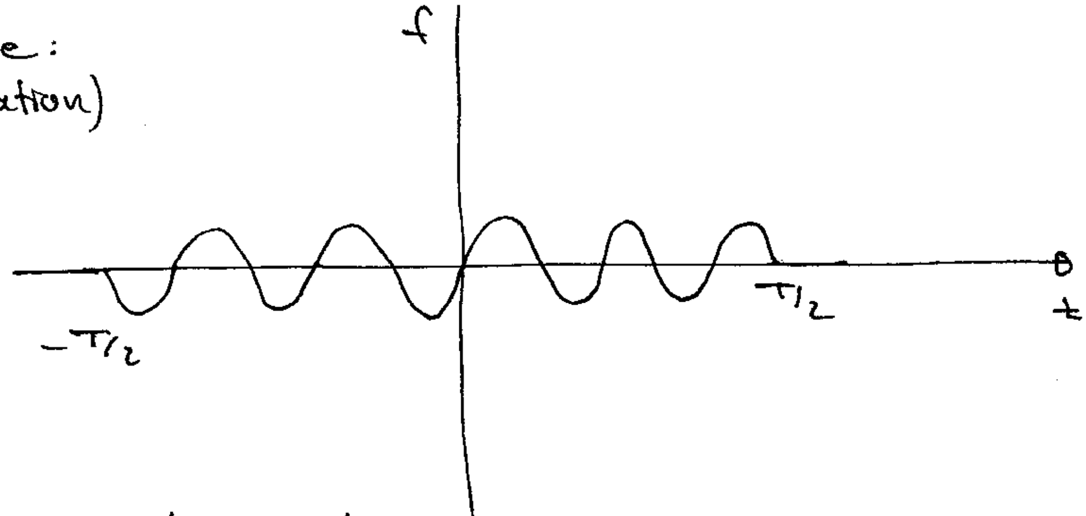
$$\boxed{\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot e^{+j\omega t}\,d\omega \\ F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\,dt \end{aligned}}$$

← Inverse transform

← Forward transform

Some Fourier Transform Pairs:

sine wave:
(finite duration)



$$f(t) = \begin{cases} \sin \omega_0 t & |t| < T/2 \\ 0 & \text{elsewhere} \end{cases}$$

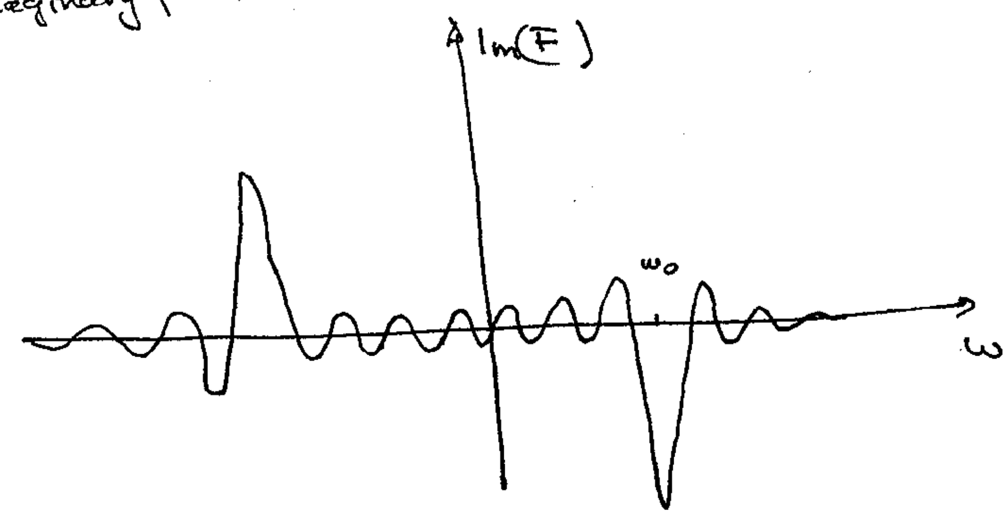
Then:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{T/2} \sin(\omega_0 t)\, e^{-j\omega t}\, dt$$

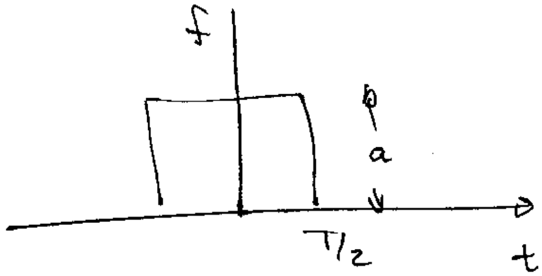
$$= -j\frac{T}{2}\left[\frac{\sin[(\omega-\omega_0)T/2]}{(\omega-\omega_0)T/2} - \frac{\sin[(\omega+\omega_0)T/2]}{(\omega+\omega_0)T/2}\right]$$

plot imaginary part:

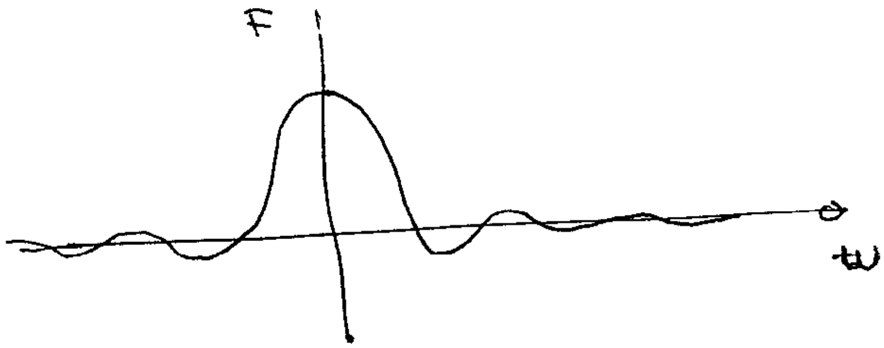


shows "energy" of wave as a function of frequency.
(energy density)

Pulse:

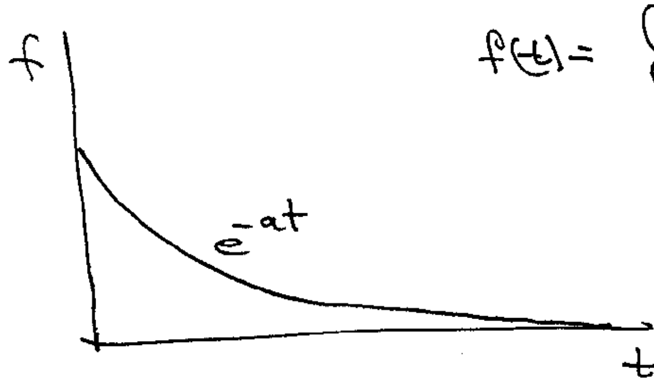


$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = aT \frac{\sin(\omega T/2)}{\omega T/2}$$

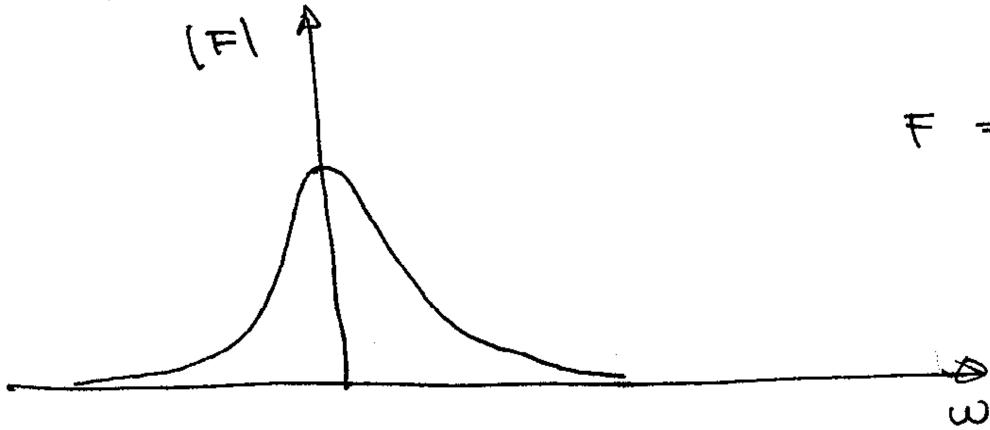


as $T \to \infty$ peak gets narrower and higher.

Decaying exponential

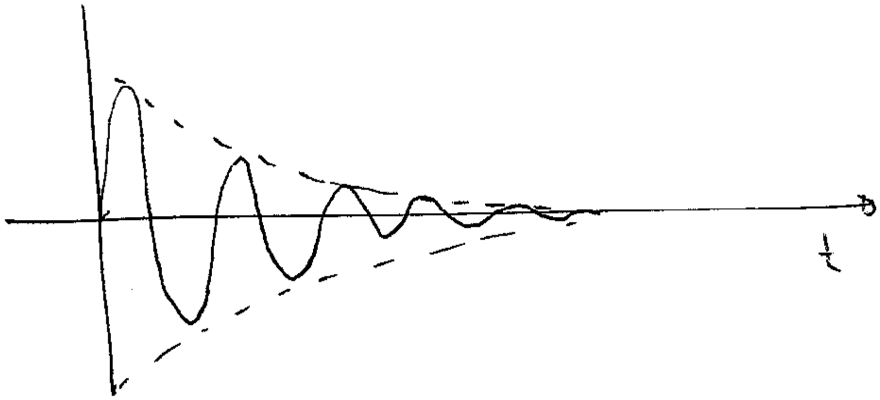


$$f(t) = \begin{cases} 0 & t<0 \\ e^{-at} & t>0 \end{cases}$$



$$F = \frac{1}{a+j\omega}$$

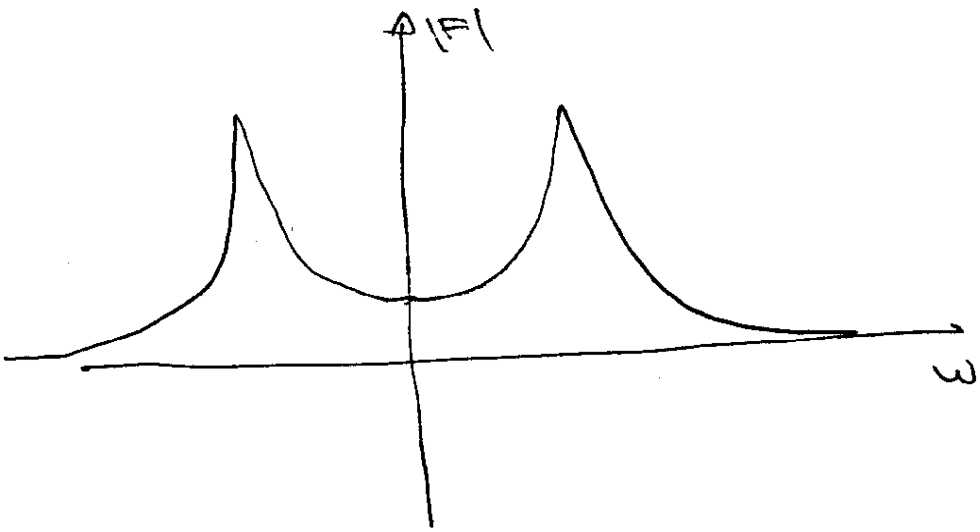
Decaying sinusoidal



$$f(t) = \begin{cases} 0 \\ e^{-at} \cdot \sin \omega_0 t \end{cases}$$

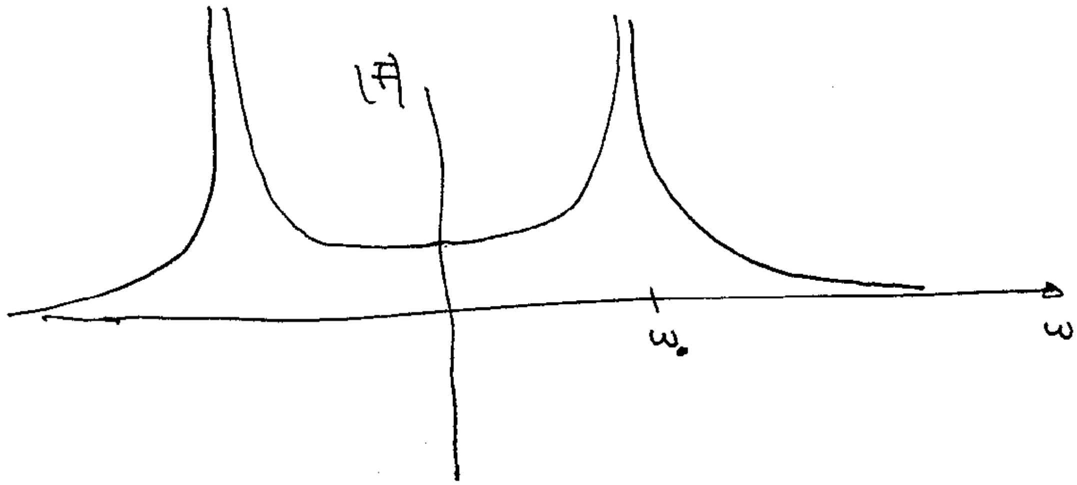
$$F(\omega) = \frac{\omega_0}{(a+j\omega)^2 + \omega_0^2} = \frac{\omega_0}{a^2 + j2a\omega - \omega^2 + \omega_0^2}$$

$$|F(\omega)| = \frac{\omega_0}{\left[(a^2 + \omega_0^2 - \omega^2)^2 + (2a\omega)^2\right]^{1/2}}$$



as $a \to 0$

$$|F(\omega)| = \frac{\omega_0}{(\omega_0^2 - \omega^2)}$$



Fourier Transform Properties

Linear $f_1(t) \to F_1(\omega)$

$f_2(t) \to F_2(\omega)$

then
$$a f_1(t) + b f_2(t) \to a F_1(\omega) + b F_2(\omega)$$

Symmetry

f even $\to$ F real

f odd $\to$ F imaginary

or $F(\omega) = \overline{F}(-\omega)$

ie $\mathrm{Re}(F(\omega)) = \mathrm{Re}(F(-\omega))$

$\mathrm{Im}(F(\omega)) = -\mathrm{Im}(F(-\omega))$

<u>Shift</u>

$$f(t) \to F(\omega)$$

$$f(t+\tau) \to e^{j\omega\tau} F(\omega)$$

Energy of a signal:

Power of a signal (or function)

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

Parseval's theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$\Rightarrow |F(\omega)|^2$ is a measure of energy of the function/freq. Energy contained in one frequency band

$$\Delta E = \frac{1}{2\pi} \int_{\omega_2}^{\omega_1} |F(\omega)|^2 d\omega$$

we call $\phi(\omega) = |F(\omega)|^2$

energy (power) density spectrum

Fourier transform of a derivative of a function

$$f(t) \to F(\omega)$$

$$\frac{d^n f(t)}{dt^n} \to (j\omega)^n F(\omega)$$

since:

$$\frac{df}{dt} \to \int_{-\infty}^{\infty} \frac{df}{dt} \cdot e^{-j\omega t} =$$

$$= \overset{0}{\cancel{f(t)\, e^{-j\omega t}}} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\,(-j\omega) \cdot e^{-j\omega t}\, dt$$

$$= j\omega\, F(\omega)$$

Linear Systems

$$f(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t) \qquad \longrightarrow$$

$$F(\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(\omega) = H(j\omega) \cdot F(\omega)$$

so $\dfrac{Y}{F} = H(j\omega)$

if f is an impulse then $F = 1$

$$\left(F = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t} = e^{-j\omega t}\Big|_{t=0} = 1 \right)$$

We then have

$$\boxed{Y = H(j\omega)}$$

so the Fourier transform of the output is equal to the system transfer function.

Also, in terms of power spectrum

$$\Phi_{yy}(\omega) = |H|^2 \, \Phi_{ff}(\omega) \qquad \text{or}$$

$$Y(\omega) \cdot Y^*(\omega) = |H|^2 \cdot F(\omega) \cdot F^*(\omega)$$

where $Y^*(\omega)$ is the complex conjugate of Y

$$|H|^2 = \frac{\Phi_{yy}}{\Phi_{ff}}$$

Also $$H(\omega) = \frac{Y(\omega)}{F(\omega)} = \frac{\text{FT of output}}{\text{FT of input}}$$