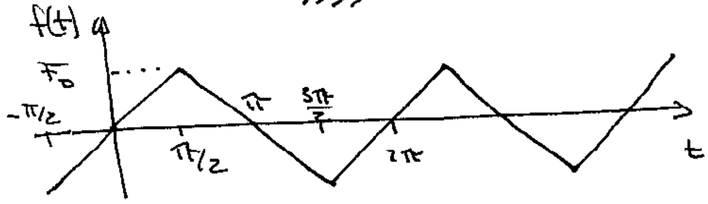
Fourier Series Example

Determine the displacement of the mass when it is excited by the force f(t)

$K=9,\ b=.3,\ m=1$



Expand f(t) using Fourier Series:

odd function $\rightarrow$ only sine terms will be present so $a_n = 0$

$$b_n = \frac{2}{T}\int_{-T/4}^{3T/4} f(t)\sin(n\omega_0 t)\,dt$$

Here $T = 2\pi$ sec $\Rightarrow T\cdot\omega_0 = 2\pi \Rightarrow \omega_0 = 1$

For $-\pi/2 < t < \pi/2 \longrightarrow f(t) = \frac{2F_0}{\pi}\cdot t$

$\pi/2 < t < 3\pi/2 \longrightarrow f(t) = 2F_0\left(1 - \frac{t}{\pi}\right)$

so $$b_n = \frac{2}{2\pi}\left\{\int_{-\pi/2}^{\pi/2}\frac{2F_0}{\pi}\cdot t\sin(nt)\,dt + \int_{\pi/2}^{3\pi/2} 2F_0\left(1-\frac{t}{\pi}\right)\sin(nt)\,dt\right\}$$

integrate by parts yields:

$$b_n = \frac{8F_0}{n^2\pi^2}\sin(n\pi/2)$$

$b_1 = \frac{8F_0}{\pi^2}$, $b_2 = 0$, $b_3 = -\frac{8F_0}{9\pi^2}$, $b_4 = 0$, $b_5 = +\frac{8F_0}{25\pi}$

so $$f(t) = \frac{8F_0}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\sin[(2m-1)t]$$

Transfer function: (from before)

$$H(i\omega) = \frac{X}{F} = \frac{1/k}{1-\left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}}$$

$$= |H(i\omega)| \cdot e^{i\theta} \qquad \text{where}$$

$$|H(i\omega)| = \frac{1/k}{\left[\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right]^{1/2}}$$

and $$\theta = -\tan^{-1}\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

Here $\omega_n = \sqrt{\frac{9}{1}} = 3$ and $\zeta = \frac{b}{2\cdot\sqrt{k\cdot m}} = 0.05$

so $$|H(i\omega)| = \frac{1/9}{\left[\left[1-\left(\frac{\omega}{3}\right)^2\right]^2 + 0.01\left(\frac{\omega}{3}\right)^2\right]^{1/2}}$$

$$\theta = -\tan^{-1}\left(\frac{0.1\left(\frac{\omega}{3}\right)}{1-\left(\frac{\omega}{3}\right)^2}\right)$$

Response:

Response to $f(t) = F \cdot \sin \omega t$ is

$$x(t) = F_0 \cdot |H(i\omega)| \cdot \sin(\omega t + \phi)$$

Response to the sawtooth is the sum of the response to each term in the Fourier series

$\frac{\omega}{\omega_n}$	b_n	$\lvert H(i\omega)\rvert$	$b_n \cdot H(i\omega)$	ϕ	response
$1/3$	$\frac{8F_0}{\pi^2} = F_0\, 0.82$	$\frac{1/9}{\left[\left(1-\left(\frac{1}{3}\right)^2\right)^2 + 0.01\left(\frac{1}{3}\right)^2\right]^{1/2}} = 0.1249$	$F_0\, 0.1024$	-0.0375	$x_1 = F_0\, 0.1024 \cdot \sin(t - 0.0375)$
$3/3$	$-F_0\, 0.090$	1.111	$-F_0\, 0.100$	$-\pi/2$	$x_2 = -F_0\, 0.100 \cdot \sin(3t - \pi/2)$
$3/5$	$F_0\, 0.0324$	0.0622	$F_0\, 0.0020$	-3.235	$x_3 = F_0\, 0.020 \cdot \sin(5t - 3.235)$
⋮	⋮	⋮	⋮	⋮	⋮

Total steady state response

$$X = X_1 + X_2 + X_3 + \cdots$$

$$= F_0\left(0.1024 \sin(t - 0.0375) - 0.100 \sin(3t - \pi/2) + 0.020 \sin(5t - 3.235) + \cdots\right)$$

Spectra