Frequency Response

$$F(t) = F_0 e^{st} \longrightarrow \boxed{H(s)} \longrightarrow y_{ss} = H(s)\cdot F_0 e^{st}$$

Since $\sin\omega t = \mathrm{Im}(e^{i\omega t})$ and
$\cos\omega t = \mathrm{Re}(e^{i\omega t})$

we use $e^{i\omega t}$ as input when the actual input is $\sin\omega t$ and/or $\cos\omega t$.

So for <u>sine and cosine</u> input:

$$\boxed{\boxed{s = i\omega}}$$

For

$$F(t) = F_0 \cos\omega t = F_0\,\mathrm{Re}(e^{i\omega t}) \longrightarrow \boxed{H} \longrightarrow y_{ss} = ?$$

we use

$$F_0 e^{i\omega t} \longrightarrow \boxed{H(i\omega)} \longrightarrow y_{ss} = H(i\omega)\cdot F_0 e^{i\omega t}$$

Actual output: $y_{ss} = \mathrm{Re}\left(H(i\omega)\, F_0 e^{i\omega t}\right)$

$$y_{ss} = H(i\omega)\cdot F_0 e^{i\omega t}$$

$$= \frac{N(i\omega)}{D(i\omega)}\cdot F_0 e^{i\omega t} = \frac{|N(i\omega)|\cdot e^{i\phi_N}}{|D(i\omega)| e^{i\phi_D}}\cdot F_0 e^{i\omega t}$$

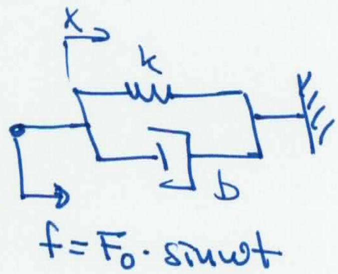
$$= \frac{|N(i\omega)|}{|D(i\omega)|}\cdot F_0 e^{i(\omega t+\phi)} \qquad \text{where } \phi = \phi_N - \phi_D$$

$$= |H(i\omega)|\cdot F_0 e^{i(\omega t+\phi)}$$

$$= |H(i\omega)|\cdot F_0 \left(\cos(\omega t+\phi) + i\sin(\omega t+\phi)\right)$$

For $\cos(\omega t)$ input keep $\cos(\omega t+\phi)$ part.
For $\sin\omega t$ input keep $\sin(\omega t+\phi)$ part.

<u>Ex</u> First order system:
Determine the steady state displacement, x.



$$b\dot{x} + kx = f \Rightarrow \tau\dot{x} + x = \frac{f}{k} \qquad \text{where } f = F_0 \cdot \sin\omega t$$

$$H(s) = \frac{X}{F_0} = \frac{1/k}{\tau s + 1} \qquad \text{for sine input:}$$

$$H(i\omega) = \frac{1/k}{i\tau\omega + 1}$$

$$x_{ss}(t) = \frac{1}{k}\frac{1}{i\tau\omega + 1}\cdot F_0 \cdot e^{i\omega t}$$

$$= \frac{1}{k}\frac{1}{\sqrt{(\tau\omega)^2+1}\cdot e^{i\phi_D}}\cdot F_0\, e^{i\omega t}$$

$$= \frac{1}{k}\frac{1}{\sqrt{(\tau\omega)^2+1}}\cdot F_0\, e^{i(\omega t+\phi)}$$

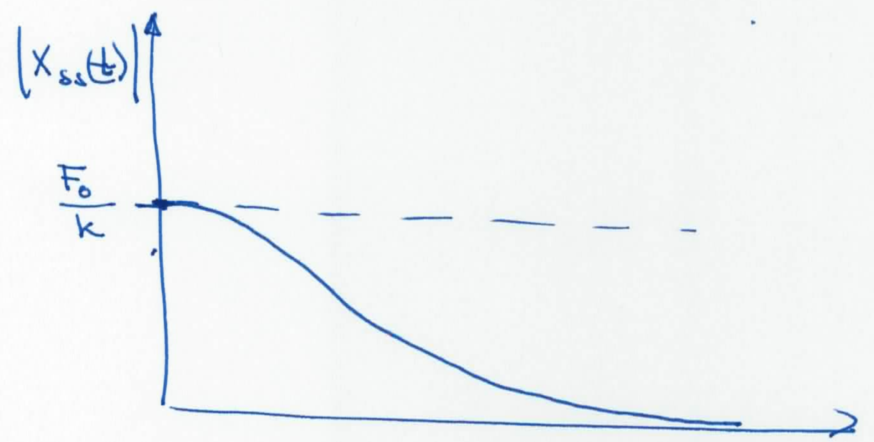
where $\phi = -\phi_D = -\tan^{-1}\left(\frac{\tau\omega}{1}\right)$

$$x_{ss}(t) = \frac{1}{k}\frac{1}{\sqrt{(\tau\omega)^2+1}}\cdot F_0\left(\cos(\omega t+\phi) + i\sin(\omega t+\phi)\right)$$

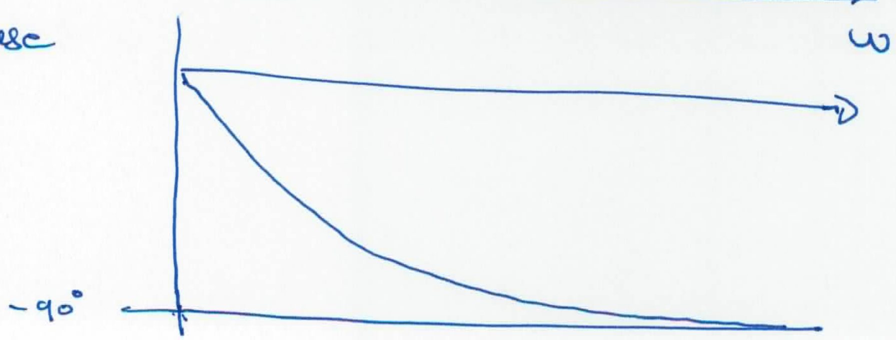
We have sine input so take imaginary part:

$$x_{ss}(t) = \frac{1}{k}\frac{1}{\sqrt{(\tau\omega)^2+1}} F_0 \sin(\omega t+\phi)$$

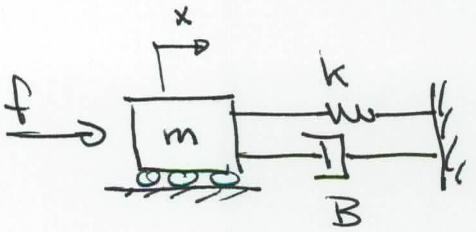
plot amplitude:



phase



Second order system



$$m\ddot{x} + kx + B\dot{x} = f = F\cdot e^{st}$$

$$ms^2 X + BsX + kX = F$$

$$H(s) = \frac{X}{F} = \frac{1}{ms^2 + Bs + k} =$$

$$= \frac{1}{m} \frac{1}{s^2 + \frac{B}{m}s + \frac{k}{m}} = \frac{1}{m} \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{B}{2\sqrt{k\cdot m}}$$

$$= \frac{1}{m\cdot\omega_n^2} \frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}\cdot s + 1}$$

$$H(s) = \frac{1}{k} \frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} s + 1}$$

For sinusoidal input:

$$H(i\omega) = \frac{1}{k} \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + i2\zeta\frac{\omega}{\omega_n}}$$

$$H(i\omega) = \frac{1}{k} \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2\right\}^{1/2}} \cdot e^{-i\phi_D}$$

For $f = Fe^{i\omega t}$ we get

$$x = \frac{1}{k}\frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2\right\}^{1/2}} e^{-i\phi_0}\cdot Fe^{i\omega t}$$

$$\Rightarrow \quad x = \frac{1}{k}\frac{F}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2\right\}^{1/2}} e^{i(\omega t+\phi)}$$

where $\phi = -\phi_0 = \tan^{-1}\left(\dfrac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$

For $f = F\sin(\omega t)$

$$x = \frac{F}{k}\frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2\right\}^{1/2}}\cdot\sin(\omega t+\phi)$$

$$x = x_{STAT}\cdot AF\,\sin(\omega t+\phi)$$

AF = amplification factor = $\dfrac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2\right\}^{1/2}}$