General Solution of Linear State Equations

From earlier you know how to obtain the state equations

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + B\underset{\sim}{u} \qquad (1)$$

$$\underset{\sim}{y} = C\underset{\sim}{x} + D\underset{\sim}{u} \qquad (2)$$

So we need to know how to solve eqn (1) so we can plug $\underset{\sim}{x}$ into (2)

Homogeneous Solution

What is the solution of

$$\begin{cases} \dot{\underset{\sim}{x}} = A\underset{\sim}{x} \\ \underset{\sim}{x}(0) = x_0 \end{cases} \quad \text{where } \underset{\sim}{u} = \underset{\sim}{0}$$

Remember how this was done when $\underset{\sim}{x} = x$ = scalar

$$\begin{cases} \dot{x} = ax \\ x(0) = x_0 \end{cases} \qquad (3)$$

Assume $x = E e^{st}$ and plug back into (3)

$$sEe^{st} = aEe^{st} \Rightarrow s = a \Rightarrow x = Ee^{at}$$

but $x = x_0$ at $t = 0$ so

$$x_0 = E\, e^{a \cdot 0} \Rightarrow E = x_0$$

so $x = x_0 \cdot e^{at}$

for the vector case:

$$\begin{cases} \dot{\underset{\sim}{x}} = A\, \underset{\sim}{x} \\ \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \end{cases} \qquad (4)$$

we get

$$\underset{\sim}{x} = e^{At} \cdot \underset{\sim}{x}_0 \qquad (5)$$

where $e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$

(just like $e^{at} = 1 + at + \frac{a^2 t^2}{2!} + \cdots$) ← scalar.

Notice that

$$\frac{d}{dt}\left(e^{At}\right) = A + A^2 t + \frac{A^3 t^2}{2!} + \cdots = A\left(I + At + \frac{A^2 t^2}{2!} + \cdots\right.$$
$$= A e^{At}$$

so $\underset{\sim}{x} = e^{At} \cdot x_0$ is a solution to (4)

check: plug in

LHS: $\frac{d}{dt}\left(e^{At} \cdot x_0\right) = \frac{d}{dt}\left(e^{At}\right) \cdot x_0 = A e^{At} \cdot x_0$

RHS: $A\underset{\sim}{x} = Ae^{At}\cdot\underset{\sim}{x}_0 \Rightarrow$ LHS = RHS

<u>Summary</u>: The solution of

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$$

$$\underset{\sim}{x}(0) = \underset{\sim}{x}_0$$

is

$$\boxed{\underset{\sim}{x} = e^{At}\cdot\underset{\sim}{x}_0}$$

The matrix $\phi(t) = e^{At}$ is called the state transition matrix.

<u>Forced Response</u>

$$\begin{cases} \dot{\underset{\sim}{x}} = A\underset{\sim}{x} + B\underset{\sim}{u} \\ \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \end{cases}$$

<u>Homogeneous part</u>: $\boxed{\underset{\sim}{x}_H = e^{At}\cdot\underset{\sim}{x}_0}$

<u>Particular solution</u>: $\boxed{\underset{\sim}{x}_P = \int_0^t e^{A(t-\tau)}\cdot B\, u(\tau)\, d\tau}$

<u>proof</u>: Let $\underset{\sim}{x}_P = e^{At}\cdot \underset{\sim}{q}(t)$ — unknown function.

Then:

LHS: $$\frac{dx_p}{dt} = \frac{d}{dt}\left(e^{At}\right)\cdot q(t) + e^{At}\,\frac{d}{dt}\left(q(t)\right)$$

$$= A\,e^{At}\cdot q(t) + e^{At}\,\frac{d}{dt}\left(q(t)\right)$$

RHS:
$$A\underset{\sim}{x} + B\underset{\sim}{u} = A e^{At}\cdot q(t) + B\cdot \underset{\sim}{u}$$

$$\text{LHS} = \text{RHS} \Rightarrow \quad e^{At}\frac{d}{dt}\left(q(t)\right) = B\cdot \underset{\sim}{u}$$

$$\Rightarrow \quad \frac{d}{dt}\left(q(t)\right) = e^{-At}B\cdot \underset{\sim}{u}$$

$$q(t) = \int_0^t e^{-At} B\, u(\tau)\, d\tau$$

$$x_p = e^{At}\cdot q(t)$$

$$x_p = e^{At}\cdot \int_0^t e^{-A\tau}\cdot B\, u(\tau)\, d\tau$$

$$x_p = \int_0^t e^{A(t-\tau)}\cdot B\cdot u(\tau)\, d\tau$$

Total response: $\underset{\sim}{x} = \underset{\sim}{x}_H + \underset{\sim}{x}_P$

$$\boxed{\underset{\sim}{x} = e^{At}\cdot \underset{\sim}{x}_0 + \int_0^t e^{A(t-\tau)} B\cdot u(\tau)\, d\tau}$$

Procedure for calculating e^{At}:

As long as we have distinct eigenvalues (all different) A can be written

$$A = M \Lambda M^{-1}$$

where $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$ where the λ's are the eigenvalues of A

and $M = [\underset{\sim}{m}_1 \; \underset{\sim}{m}_2 \; \underset{\sim}{m}_3 \cdots \underset{\sim}{m}_n]$

where the $\underset{\sim}{m}$'s are the eigen vectors of A.

so since $A = M \Lambda M^{-1}$

we get.

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \ldots$$

$$e^{At} = I + M\Lambda M^{-1} + \frac{(M\Lambda \cancel{M^{-1}})(\cancel{M}\Lambda M^{-1})t^2}{2!} + \frac{(M\Lambda \cancel{M^{-1}})(\cancel{M}\Lambda \cancel{M^{-1}})(\cancel{M}\Lambda M^{-1})}{3!} + \ldots$$

$$e^{At} = M\left(I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + \dots\right)M^{-1}$$

$$e^{At} = M\left[\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ 0 & & \ddots \; \lambda_n^2 \end{bmatrix} + \dots\right]M^{-1}$$

$$e^{At} = M\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix}M^{-1}$$

$$\boxed{e^{At} = M \cdot e^{\Lambda t} \cdot M^{-1}}$$

where

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{pmatrix}$$

Uncoupling of the state equations (7)

consider $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$

Change of variable: $x = M \cdot x'$

Then

$$M\dot{x}' = AMx'$$

$$\dot{x}' = M^{-1}AMx'$$

But $A = M\Lambda M^{-1}$ or $\Lambda = M^{-1}AM$

so

$$\dot{x}' = \Lambda x'$$

or

$$\begin{pmatrix} \dot{x}'_1 \\ \dot{x}'_2 \\ \vdots \\ \dot{x}'_n \end{pmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{matrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{matrix}$$

Equations are uncoupled!

solve

$$\begin{aligned} x'_1 &= E_1 e^{\lambda_1 t} \\ x'_2 &= E_2 e^{\lambda_2 t} \\ &\vdots \\ x'_n &= E_n e^{\lambda_n t} \end{aligned} \quad \Rightarrow \quad \underset{\sim}{x}' = \begin{pmatrix} E_1 e^{\lambda_1 t} \\ E_2 e^{\lambda_2 t} \\ \vdots \\ E_n e^{\lambda_n t} \end{pmatrix}$$