Energy Methods

Conservation of energy:

If we have no losses in the system "mechanical" energy is conserved:

$$T + V = \text{constant}$$

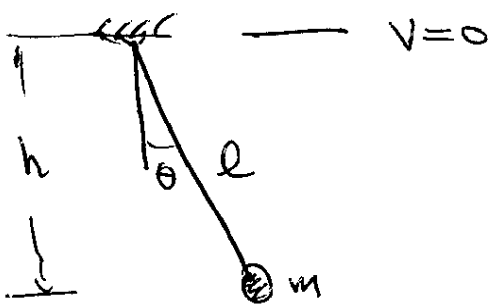
T – kinetic energy

V – Potential energy

Then $\frac{d}{dt}(T+V) = 0$

"no change with time"

Ex Pendulum:



Potential Energy: $V = -mgh = -mg\,l\cos\theta$

Kinetic Energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2$

so $\frac{d}{dt}\left(\frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta\right) = 0$

$$ml^2 \dot{\theta}\ddot{\theta} + mgl\sin\theta\,\dot{\theta} = 0$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

The system is nonlinear because of the $\sin\theta$ term

For small motions, $|\theta| << 1$

then $\sin\theta \sim \theta$

so $\ddot{\theta} + \frac{g}{l}\theta = 0$

so $\omega_n = \sqrt{\frac{g}{l}}$

Lagrange Equation

Define $L = T - V$ - the Lagrangian

The the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k} = Q_k \qquad k = 1, \dots, n$$

where $Q_k = \sum_{i=1}^{N} \underset{\sim}{F_i} \cdot \frac{\partial \underset{\sim}{r_i}}{\partial x_k}$

where $\underset{\sim}{F_i}$ = nonconservative forces acting on the i^{th} particle (mass)

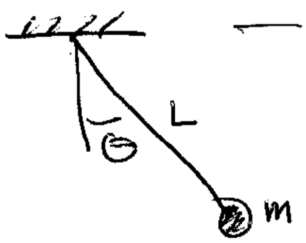
$\underset{\sim}{r_i} = \underset{\sim}{r_i}(x_1, x_2, x_3 \dots, t)$ Position vector to the i^{th} particle

x_k = the generalized coordinate

N is the total number of masses

n is the degree of freedom

Ex



Here $x_1 = \theta$, $n=1$ (one degree of freedom)

$$V = -mgL\cos\theta$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(L\dot{\theta})^2$$

$$L = T - V = \frac{1}{2}m(L\dot{\theta})^2 + mgL\cos\theta$$

so Lagrange equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

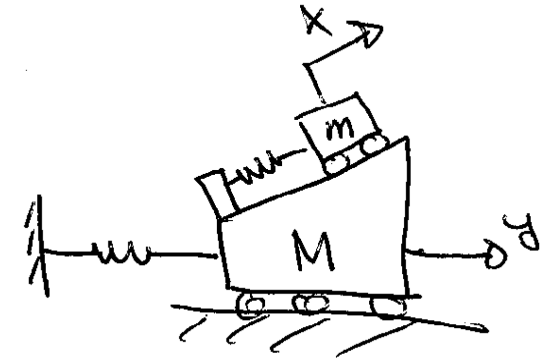
$$\frac{d}{dt}(mL^2\dot{\theta}) - mgL(-\sin\theta) = 0$$

$$mL^2\ddot{\theta} + mgL\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \quad ✓$$

or $$\ddot{\theta} + \frac{g}{L}\theta = 0 \quad \text{for } |\theta| \ll 1$$

Ex 2



No gravity, $g = 0$

Here $x_1 = y$, $x_2 = x$, $n = 2$

$$V = \frac{1}{2} k_1 \cdot y^2 + \frac{1}{2} k_2 (x - y\cos\theta)^2$$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} M \dot{y}^2$$

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} M \dot{y}^2 - \frac{1}{2} k_1 y^2 - \frac{1}{2} k_2 (x - y\cos\theta)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \implies$$

$$\frac{d}{dt}(M \cdot \dot{y}) + k_1 y + k_2 (x - y\cos\theta)(-\cos\theta) = 0$$

$$\implies \boxed{M\ddot{y} + (k_1 + k_2 \cos^2\theta) \cdot y - k_2 \cos\theta \, x = 0} \quad (1) \Leftarrow$$

Also

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial \dot{x}} = 0$$

$$\frac{d}{dt}(m \dot{x}) + k_2 (x - y\cos\theta) \cdot 1 = 0$$

$$\boxed{m\ddot{x} + k_2 x - k_2 \cos\theta \cdot y = 0} \quad (2) \Leftarrow$$

or in matrix form, Equ (1) and (2) $\implies$

$$\begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{y} \\ \ddot{x} \end{pmatrix} + \begin{bmatrix} k_1 + k_2 \cos^2\theta & k_2 \cos\theta \\ -k_2 \cos\theta & k_2 \end{bmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$