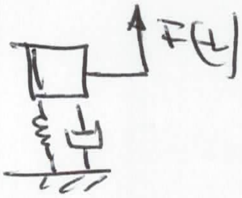
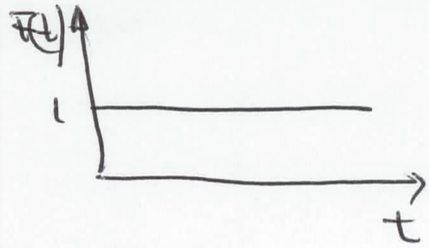
Ex3





①

$m = 2\,kg$, $k = 8\ N/m$, $B = 1\ Ns/m$

What is the response if initial displacement is $y_0 = 0.1\ m$ (UP) and initial velocity is $v_m(0) = 0$.

State Equation:

$$\begin{pmatrix} \dot{v}_m \\ \dot{f}_k \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 \\ 8 & 0 \end{pmatrix} \begin{pmatrix} v_m \\ f_k \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} F$$

Initial condition: $y(0) = 0.1\ m \Rightarrow$

$$f_k(0) = k\,y(0) \Rightarrow f_k(0) = 8 \cdot 0.1 = 0.8\ N.$$

so

$$\begin{pmatrix} v(0) \\ f_k(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0.8 \end{pmatrix}$$

Eigenvalues:

$$\begin{vmatrix} -1/2 - \lambda & -1/2 \\ 8 & -\lambda \end{vmatrix} = 0 \Rightarrow$$

$$(-1/2 - \lambda)(-\lambda) + 4 \Rightarrow \lambda^2 + \frac{1}{2}\lambda + 4 = 0$$

$$\lambda = -\frac{1}{4} \pm \sqrt{\frac{1}{16} - 4} = -\frac{1}{4} \pm i\frac{\sqrt{63}}{4}$$

<u>Eigenvectors</u>

$\lambda_1 = -\frac{1}{4} + i\frac{\sqrt{63}}{4}$:

$$\begin{pmatrix} -\frac{1}{2} - \left(-\frac{1}{4} + i\frac{\sqrt{63}}{4}\right) & -\frac{1}{2} \\ 8 & 0 - \left(-\frac{1}{4} + i\frac{\sqrt{63}}{4}\right) \end{pmatrix} \begin{pmatrix} m^1_1 \\ m^1_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let $m^1_1 = 1$ then

$$-\frac{1}{2} - \left(-\frac{1}{4} + i\frac{\sqrt{63}}{4}\right)\cdot 1 - \frac{1}{2} m^1_2 = 0$$

$$m^1_2 = -\frac{1}{2}\left(1 + i\sqrt{63}\right)$$

so $$\underset{\sim}{m}^1 = \begin{pmatrix} 1 \\ -\frac{1}{2}\left(1 + i\sqrt{63}\right) \end{pmatrix}$$

Also: $\lambda_2 = \frac{1}{4} - i\frac{\sqrt{63}}{4}$

$$\Rightarrow \underset{\sim}{m}^2 = \begin{pmatrix} 1 \\ -\frac{1}{2}\left(1 - i\sqrt{63}\right) \end{pmatrix}$$

<u>Hence:</u>

$$M = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2}\left(1 + i\sqrt{63}\right) & -\frac{1}{2}\left(1 - i\sqrt{63}\right) \end{bmatrix}$$

and

$$e^{\Lambda t} = \begin{bmatrix} e^{\left(-\frac{1}{4} + i\frac{\sqrt{63}}{4}\right)\cdot t} & 0 \\ 0 & e^{\left(-\frac{1}{4} - i\frac{\sqrt{63}}{4}\right)\cdot t} \end{bmatrix}$$

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e^{At} :

$$e^{At} = M \cdot e^{\Lambda t} \cdot M^{-1} =$$

$$= \begin{bmatrix} 1 & 1 \\ -\frac{1}{2}(1+i\sqrt{63}) & -\frac{1}{2}(1-i\sqrt{63}) \end{bmatrix} \begin{bmatrix} e^{\left(-\frac{1}{4}+i\frac{\sqrt{63}}{4}\right)t}, & 0 \\ 0, & e^{\left(-\frac{1}{4}-i\frac{\sqrt{63}}{4}\right)t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}(1-i\sqrt{63}) & -1 \\ \frac{1}{2}(1+i\sqrt{63}) & 1 \end{bmatrix} \frac{1}{i\sqrt{63}}$$

$$e^{At} = \frac{e^{-t/4}}{i\sqrt{63}} \begin{bmatrix} e^{i\frac{\sqrt{63}}{4}t} & e^{-i\frac{\sqrt{63}}{4}t} \\ -\frac{1}{2}(1+i\sqrt{63})e^{i\frac{\sqrt{63}}{4}t}, & -\frac{1}{2}(1-i\sqrt{63})e^{-i\frac{\sqrt{63}}{4}t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}(1-i\sqrt{63}) & -1 \\ \frac{1}{2}1+i\sqrt{63} & 1 \end{bmatrix}$$

$$e^{At} = \frac{e^{-t/4}}{i\sqrt{63}} \begin{bmatrix} -\frac{1}{2}(1-i\sqrt{63})e^{i\frac{\sqrt{63}}{4}t} + \frac{1}{2}(1+i\sqrt{63})e^{-i\frac{\sqrt{63}}{4}t}, & -e^{i\frac{\sqrt{63}}{4}t} + e^{-i\frac{\sqrt{63}}{4}t} \\ \frac{1}{4}(1+63)e^{i\frac{\sqrt{63}}{4}t} - \frac{1}{4}(1+63)e^{-i\frac{\sqrt{63}}{4}t}, & \frac{1}{2}(1+i\sqrt{63})e^{i\frac{\sqrt{63}}{4}t} - \frac{1}{2}(1-i\sqrt{63})e^{-i\frac{\sqrt{63}}{4}t} \end{bmatrix}$$

$$e^{At} = \frac{e^{-t/4}}{i\sqrt{63}} \begin{bmatrix} i\sqrt{63}\cos\frac{\sqrt{63}}{4}t - i\sin\frac{\sqrt{63}}{4}t, & -2i\sin\frac{\sqrt{63}}{4}t \\ 32i\sin\frac{63}{4}t, & i\sqrt{63}\cos\frac{\sqrt{63}}{4}t + i\sin\frac{\sqrt{63}}{4}t \end{bmatrix}$$

since $e^{i\frac{\sqrt{63}}{4}\cdot t} = \cos\frac{\sqrt{63}}{4}t + i\sin\frac{\sqrt{63}}{4}t$

$e^{-i\frac{\sqrt{63}}{4}t} = \cos\frac{\sqrt{63}}{4}t - i\sin\frac{\sqrt{63}}{4}\cdot t$

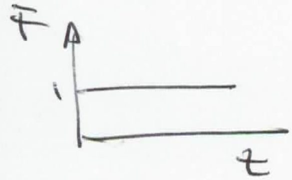
finally:

$$e^{At} = e^{-t/4}\begin{bmatrix} \cos\frac{\sqrt{63}}{4}\cdot t - \frac{1}{\sqrt{63}}\sin\frac{\sqrt{63}t}{4}, & \frac{-2}{\sqrt{63}}\sin\frac{\sqrt{63}}{4}\cdot t \\ \frac{32}{\sqrt{63}}\sin\frac{\sqrt{63}}{4}t, & \cos\frac{\sqrt{63}}{4}\cdot t + \frac{1}{\sqrt{63}}\sin\frac{\sqrt{63}}{4}\cdot t \end{bmatrix}$$

solution:

$$\begin{pmatrix} V_m \\ f_k \end{pmatrix} = e^{At}\cdot\begin{pmatrix} V_m \\ f_k \end{pmatrix}_0 + \int_0^t e^{A(t-\tau)}\cdot B\,u(\tau)\,d\tau$$

But $B\,u(\tau) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$ since $F(t) = 1$



$$\begin{pmatrix} V_m \\ f_k \end{pmatrix} = e^{At}\begin{pmatrix} V_m \\ f_k \end{pmatrix}_0 + \int_0^t e^{A(t-\tau)}\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} d\tau$$

$$\begin{pmatrix} V_m \\ f_k \end{pmatrix} = e^{At}\begin{pmatrix} V_m \\ f_k \end{pmatrix}_0 - A^{-1}e^{A(t-\tau)}\Big|_0^t\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} V_m \\ f_k \end{pmatrix} = e^{At}\begin{pmatrix} V_m \\ f_k \end{pmatrix}_0 - A^{-1}\left(I - e^{At}\right)\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

Now plug in e^{At} and A^{-1}:

$$\begin{pmatrix} V_m \\ f_n \end{pmatrix} = e^{-t/4}\begin{bmatrix} \frac{-1.6}{4a}\sin at \\ 0.8\cos at + \frac{0.8}{4a}\sin at \end{bmatrix} +$$

$$-\begin{pmatrix} -1/2 & -1/2 \\ 8 & 0 \end{pmatrix}^{-1}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^{-t/4}\begin{pmatrix} \cos at - \frac{1}{4a}\sin at, & -\frac{2}{4a}\sin at \\ \frac{32}{4a}\sin at, & \cos at + \frac{1}{4a}\sin at \end{pmatrix}\right]\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$$= -\frac{1}{4}\begin{pmatrix} 0 & 1/2 \\ -8 & -1/2 \end{pmatrix}\begin{bmatrix} \frac{1}{2}\left(1 - e^{-t/4}\left(\cos at - \frac{1}{4a}\sin at\right)\right) \\ -e^{-t/4}\frac{32}{4a}\sin at \end{bmatrix} + e^{-t/4}\begin{bmatrix} \frac{-1.6}{4a}\sin at \\ 0.8\cos at + \frac{0.8}{4a}\sin at \end{bmatrix}$$

$$= \begin{bmatrix} +e^{-t/4}\frac{1}{a}\sin at \\ +\left(1 - e^{-t/4}\left(\cos at - \frac{1}{4a}\sin at\right)\right) - e^{-t/4}\frac{1}{a}\sin at \end{bmatrix} + e^{-t/4}\begin{pmatrix} \frac{-1.6}{4a}\sin at \\ 0.8\cos at + \frac{0.8}{4a}\sin at \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} V_m \\ f_n \end{pmatrix} = e^{-t/4}\begin{bmatrix} +\left(\frac{1}{a} + \frac{1.6}{a}\right)\sin at \\ (-1+0.8)\cos at + \left(\frac{1}{a} + \frac{1}{4a} + \frac{0.8}{4a}\right)\sin at \end{bmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{where } a = \frac{\sqrt{63}}{4}}$$