Lagrange equations (from Wikipedia)

This is a derivation of the Lagrange equations. This derivation is obviously above and beyond the scope of this class. Hence this is only for the very curious student. There are many classical references that one can use to get more information about this topic:

Goldstein, H. *Classical Mechanics,* second edition, (Addison-Wesley, 1980) or

Meirovitch, L. *Principles and Techniques of Vibrations*, (Prentice Hall, 1997)

Consider a single particle with mass *m* and position vector $\mathbf{r}$, moving under an applied force, $\mathbf{F}$, which can be expressed as the gradient of a scalar potential energy function $V(\mathbf{r}, t)$:

$$\mathbf{F} = -\nabla V.$$

Such a force is independent of third- or higher-order derivatives of $\mathbf{r}$, so Newton's second law forms a set of 3 second-order ordinary differential equations. Therefore, the motion of the particle can be completely described by 6 independent variables, or *degrees of freedom*. An obvious set of variables is $\{\mathbf{r}_j, \dot{\mathbf{r}}_j | j = 1, 2, 3\}$, the Cartesian components of $\mathbf{r}$and their time derivatives, at a given instant of time (i.e. position (x,y,z) and velocity (v_x,v_y,v_z)).

More generally, we can work with a set of generalized coordinates, q_j, and their time derivatives, the generalized velocities, $\dot{q}_j$. The position vector, $\mathbf{r}$, is related to the generalized coordinates by some *transformation equation*:

$$\mathbf{r} = \mathbf{r}(q_i, q_j, q_k, t).$$

For example, for a simple pendulum of length ℓ, a logical choice for a generalized coordinate is the angle of the pendulum from vertical, θ, for which the transformation equation would be

$$\mathbf{r}(\theta, \dot{\theta}, t) = (\ell \sin\theta, \ell \cos\theta).$$

The term "generalized coordinates" is really a holdover from the period when Cartesian coordinates were the default coordinate system.

Consider an arbitrary displacement $\delta \mathbf{r}$ of the particle. The work done by the applied force $\mathbf{F}$ is $W = \mathbf{F} \cdot \delta \mathbf{r}$. Using Newton's second law, we write:

$$\mathbf{F} \cdot \delta \mathbf{r} = m\ddot{\mathbf{r}} \cdot \delta \mathbf{r}.$$

Since work is a physical scalar quantity, we should be able to rewrite this equation in terms of the generalized coordinates and velocities. On the left hand side,

$$\mathbf{F} \cdot \delta \mathbf{r} = -\nabla V \cdot \sum_i \frac{\partial \mathbf{r}}{\partial q_i} \delta q_i$$

$$= -\sum_{i,j} \frac{\partial V}{\partial r_j} \frac{\partial r_j}{\partial q_i} \delta q_i$$

$$= -\sum_i \frac{\partial V}{\partial q_i} \delta q_i.$$

On the right hand side, carrying out a change of coordinates[clarification needed], we obtain:

$$m\ddot{\mathbf{r}} \cdot \delta \mathbf{r} = m \sum_{i,j} \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j$$

Rearranging slightly:

$$m\ddot{\mathbf{r}} \cdot \delta \mathbf{r} = m \sum_j \left[\sum_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \right] \delta q_j$$

Now, by performing an "integration by parts" transformation, with respect to t:

$$m\ddot{\mathbf{r}} \cdot \delta \mathbf{r} = m \sum_j \left[\sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{r}_i \frac{\partial r_i}{\partial q_j} \right) - \dot{r}_i \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial r_i}{\partial q_j} \right) \right] \right] \delta q_j$$

Recognizing that $\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial r_j}{\partial q_i} = \frac{\partial \dot{r}_j}{\partial q_i}$ and $\frac{\partial r_j}{\partial q_i} = \frac{\partial \dot{r}_j}{\partial \dot{q}_i}$, we obtain:

$$m\ddot{\mathbf{r}} \cdot \delta \mathbf{r} = m \sum_j \left[\sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \right) - \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_j} \right] \right] \delta q_j$$

Now, by changing the order of differentiation, we obtain:

$$m\ddot{\mathbf{r}} \cdot \delta\mathbf{r} = m \sum_j \left[\sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \dot{r}_i^{\,2} \right) - \frac{\partial}{\partial q_j} \left(\frac{1}{2} \dot{r}_i^{\,2} \right) \right] \right] \delta q_j$$

Finally, we change the order of summation:

$$m\ddot{\mathbf{r}} \cdot \delta\mathbf{r} = \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m \dot{r}_i^{\,2} \right) - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m \dot{r}_i^{\,2} \right) \right] \delta q_j$$

Which is equivalent to:

$$m\ddot{\mathbf{r}} \cdot \delta\mathbf{r} = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \right] \delta q_i$$

where $T = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ is the kinetic energy of the particle. Our equation for the work done becomes

$$\sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial (T - V)}{\partial q_i} \right] \delta q_i = 0.$$

However, this must be true for *any* set of generalized displacements δq_i, so we must have

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial (T - V)}{\partial q_i} \right] = 0$$

for *each* generalized coordinate δq_i. We can further simplify this by noting that V is a function solely of **r** and t, and **r** is a function of the generalized coordinates and t. Therefore, V is independent of the generalized velocities:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial V}{\partial \dot{q}_i} = 0.$$

Inserting this into the preceding equation and substituting $L = T - V$, called the Lagrangian, we obtain Lagrange's equations:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

There is one Lagrange equation for each generalized coordinate q_i. When $q_i = r_i$ (i.e. the generalized coordinates are simply the Cartesian coordinates), it is straightforward to check that Lagrange's equations reduce to Newton's second law.

The above derivation can be generalized to a system of N particles. There will be 6N generalized coordinates, related to the position coordinates by 3N transformation equations. In each of the 3N Lagrange equations, T is the total kinetic energy of the system, and V the total potential energy.

In practice, it is often easier to solve a problem using the Euler–Lagrange equations than Newton's laws. This is because not only may more appropriate generalized coordinates q_i be chosen to exploit symmetries in the system, but constraint forces are replaced with simpler relations