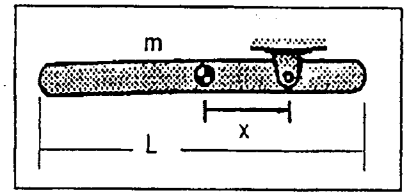
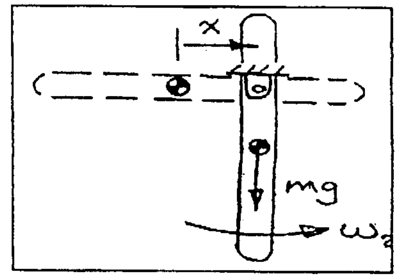
ME230 HW7 Solutions

(1) The slender bar is released from rest in the position shown. (a) Use conservation of energy to determine the angular velocity when the bar is vertical. (b) For what value of x is the angular velocity determined in part (a) a maximum?





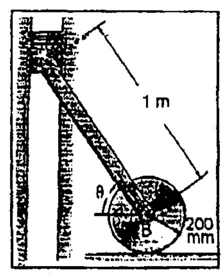
Solution Choose the datum at the level of the pin. From the conservation of energy, $T + V = const.$, where T is the kinetic energy and V is the total potential energy (see Eq 8.19). By definition, $V_1 = 0$ at the datum, and $T_1 = 0$ at the datum because the bar is released from rest there, from which $T_2 + V_2 = 0$ at any other position. The change in total potential energy in terms of the work done is $U = V_1 - V_2 = -V_2$ (see equation preceding Eq (8.19)). The work done is $U = \int_{s_1}^{s_2} F\, ds = -\int_0^{-x} mg ds = mgx = -V_2$. The kinetic energy is $T_2 = \frac{1}{2} I\omega_2^2$, where $I = \frac{mL^2}{12} + mx^2$, from which $-mgx + \left(\frac{1}{2}\right)\left(\frac{mL^2}{12} + mx^2\right)\omega_2^2 = 0$. (a) Solve

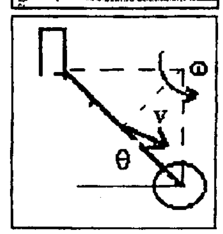
$$\omega_2^2 = \frac{2gx}{\left(\frac{1}{12}L^2 + x^2\right)},\quad \omega_2 = \sqrt{\frac{2gx}{\left(\frac{1}{12}L^2 + x^2\right)}}$$

(b) Take the derivative to find the maximum:

$$\frac{d\omega_2^2}{dx} = 0 = \frac{2g}{\left(\frac{1}{12}L^2 + x^2\right)} - \frac{2gx}{\left(\frac{1}{12}L^2 + x^2\right)^2}(2x) = 0,$$ from which $\boxed{x = \frac{L}{\sqrt{12}}}$

(2) The 4 kg slender bar is pinned to a 2 kg slider at A and to a 4 kg homogenous cylindrical disk at B. Neglect the friction force on the slider and assume that the disk rolls. If the system is released from rest with $\theta = 60^o$, what is the bar's angular velocity when $\theta = 0$?





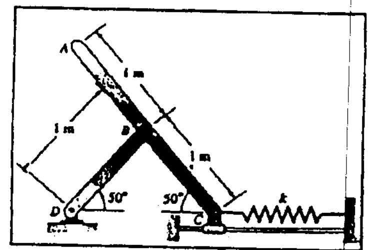
Solution Choose the datum at $\theta = 0$. The instantaneous center of the bar has the coordinates $(L\cos\theta,\ L\sin\theta)$ (see figure), and the distance from the center of mass of the bar is $\frac{L}{2}$, from which the angular velocity about the bar's instantaneous center is $v = \left(\frac{L}{2}\right)\omega$, where v is the velocity of the center of mass. The velocity of the slider is $v_A = \omega L\cos\theta$, and the velocity of the disk is $v_B = \omega L\sin\theta$ The potential energy of the system is $V_1 = m_A gL\sin\theta_1 + mg\left(\frac{L}{2}\right)\sin\theta_1$. At the datum, $V_2 = 0$. The kinetic energy is

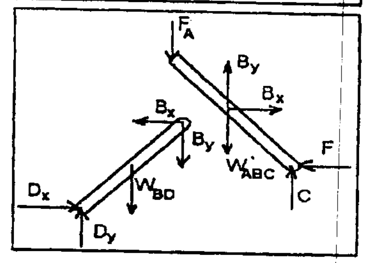
$$T_2 = \left(\frac{1}{2}\right)m_A v_A^2 + \left(\frac{1}{2}\right)mv^2 + \left(\frac{1}{2}\right)\left(\frac{mL^2}{12}\right)\omega^2 + \left(\frac{1}{2}\right)m_B v_B^2 + \left(\frac{1}{2}\right)\left(\frac{m_B R^2}{2}\right)\left(\frac{v_B}{R}\right)^2,$$ where at the datum

$v_A = \omega L\cos 0^o = \omega L,\ v_B = \omega L\sin 0^o = 0,\ v = \omega\left(\frac{L}{2}\right)$. From the conservation of energy: $V_1 = T_2$.

Solve: $\boxed{\omega = 4.515\ rad/s}$

(3) The system is in equilibrium in the position shown. The mass of the slender bar ABC is 6 *kg*, the mass of the bar BD is 3 *kg*, and the mass of the slider at C is 1 *kg*. The spring constant is $k = 200\ N/m$. If a constant 100 *N* downward force is applied at A, what is the angular velocity of the bar ABC when it has rotated 20^o from its initial position?





Solution Choose a coordinate system with the origin at D and the *x* axis parallel to DC. The equilibrium conditions for the bars: for bar BD, $\sum F_x = -B_x + D_x = 0$,

$\sum F_y = -B_y - m_{BD}g + D_y = 0$.

$\sum M_D = B_x \sin 50^o - \left(B_y + \frac{m_{BD}g}{2}\right)\cos 50^o = 0$. For the bar ABC, $\sum F_x = B_x - F = 0$, $\sum F_y = -F_A + C - m_{ABC}g + B_y = 0$.

$\sum M_C = (2F_A - B_y + m_{ABC}g)\cos 50^o - B_x \sin 50^o = 0$. At the initial position $F_A = 0$. The solution: $B_x = 30.87\ N$, $D_x = 30.87\ N$, $B_y = 22.07\ N$, $D_y = 51.5\ N$, $F = 30.87\ N$, $C = 36.79\ N$. [*Note:* Only the value $F = 30.87\ N$ is required for the purposes of this problem.] The initial stretch of the spring is $S_1 = \frac{F}{k} = \frac{30.87}{200} = 0.154\ m$. The distance D to C is $2\cos\theta$, so that the final stretch of the spring is $S_2 = S_1 + \left(2\cos 30^o - 2\cos 50^o\right) = 0.601\ m$. From the principle of work and energy: $U = T_2 - T_1$, where $T_1 = 0$ since the system starts from rest. The work done is $U = U_{force} + U_{ABC} + U_{BD} + U_{spring}$. The height of the point A is $2\sin\theta$, so that the change in height is $h = 2\left(\sin 50^o - \sin 50^o\right)$, and the work done by the applied force is $U_{force} = \int_0^h F_A dh = 100(2\sin 50^o - 2\sin 30^o) = 53.2\ N\text{-}m$. The height of the center of mass of bar BD is $\frac{\sin\theta}{2}$, so that the work done by the weight of bar BD is

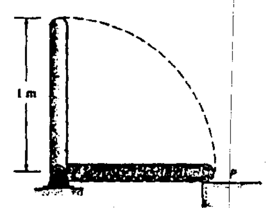
$U_{BD} = \int_0^h -m_{BD}g dh = -\frac{m_{BD}g}{2}\left(\sin 30^o - \sin 50^o\right) = 3.91\ N\text{-}m$.

The height of the center of mass of bar ABC is $\sin\theta$, so that the work done by the weight of bar ABC is $U_{ABC} = \int_0^h -m_{ABC}g dh = -m_{ABC}g\left(\sin 30^o - \sin 50^o\right) = 15.66\ N\text{-}m$. The work done by the spring is $U_{spring} = \int_{S_1}^{S_2} -ks ds = -\frac{k}{2}(S_2^2 - S_1^2) = -33.72\ N\text{-}m$. Collecting terms, the total work: $U = 39.07\ N\text{-}m$. The bars form an isosceles triangle, so that the changes in angle are equal; by differentiating the changes, it follows that the angular velocities are equal. The distance D to C is $x_{DC} = 2\cos\theta$, from which $v_C = -2\sin\theta\ \omega$, since D is a stationary. The kinetic energy is

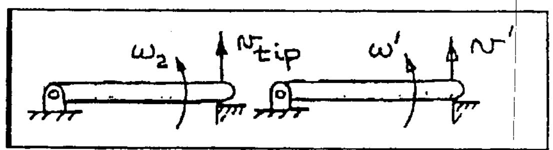
$T_2 = \left(\frac{1}{2}\right)I_{BD}\omega^2 + \left(\frac{1}{2}\right)I_{ABC}\omega^2 + \left(\frac{1}{2}\right)m_{ABC}v_{ABC}^2 + \left(\frac{1}{2}\right)m_C v_C^2 = 5\omega^2$, where $I_{BD} = \frac{m_{BD}}{3}\left(1^2\right)$, $I_{ABC} = \frac{m_{ABC}}{12}\left(2^2\right)$, $v_{ABC} = (1)\omega$, $v_C = -2\sin\theta\ \omega$. Substitute into $U = T_2$ and solve:

$\boxed{\omega = 2.795\ rad/s}$

(4) The 2 *kg* slender bar starts from rest in the vertical position and falls, striking the smooth surface at P. The coefficient of restitution of the impact is $e = 0.5$. When the bar rebounds, through what angle relative to the horizontal will it rotate? *Strategy*: Use the coefficient of restitution to relate the bar's velocity at P just after the impact to its value just before the impact.



Solution Choose a coordinate system with the origin at the center of mass of the bar at the initial position, and the y axis positive upward. The strategy is to use the principle of work and energy to determine the angular velocity of the bar the instant before impact; use the coefficient of restitution to determine the bar's velocity the instant after impact, and use the principle of work and energy to determine the maximum angle reached after rebound. From the principle of work and energy, $U = T_2 - T_1$, where $T_1 = 0$, since the bar starts from rest. The work done by gravity is



$$U = \int_{h_1}^{h_2} -mgdh = -mg\left(-\frac{L}{2} - 0\right) = mg\left(\frac{L}{2}\right) \quad N\text{-}m$$

The kinetic energy is $T_2 = \frac{1}{2} I \omega_2^2 = \frac{1}{2}\left(\frac{mL^2}{3}\right)\omega_2^2$. Substitute into $U = T_2$, to obtain

$mg\left(\frac{L}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{mL^2}{3}\right)\omega_2^2$, from which $\omega_2 = -\sqrt{\frac{3g}{L}} = -5.42 \ rad/s$. The linear velocity at the end of the bar is $v_{tip} = L\omega = -5.42 \ m/s$. From the definition of the coefficient of restitution (see Eq (8.46)), since the surface P is stationary the upward velocity at the end of the bar after rebound is

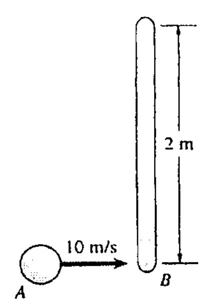
$v' = -(0.5)(v_{tip}) = 2.71 \ m/s$, and the angular velocity after rebound is $\omega' = \frac{v'}{L} = 2.71 \ rad/s$. From the principle of work and energy $U = T_2 - T_1$, where $T_2 = 0$ since the bar comes to rest at the maximum angle. The work done is $U = \int_{h_1}^{h_2} -mg\, dh = -mg(h_2 - 0) = -mgh_2$. The kinetic energy is

$T_1 = \left(\frac{1}{2}\right) I\omega'^2 = \left(\frac{1}{2}\right)\left(\frac{mL^2}{3}\right)\omega'^2$. Substitute into $U = -T_1$ and solve: $h_2 = \frac{L}{6g}\omega'^2 = 0.125 \ m$, which is the maximum height of the center of mass of the bar after rebound. The associated angle is

$$\boxed{\theta_{max} = \sin^{-1}\left(\frac{2h_2}{L}\right) = 14.48^\circ}$$

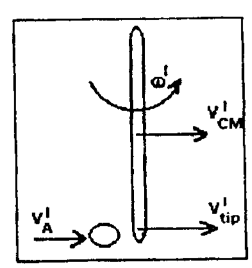
(5) The 1 *kg* sphere is traveling at 10 *m/s* when it strikes the and of the 4 *kg* stationary slender bar B. If the sphere adheres to the bar, what is the bar's angular velocity after the impact?

Solution The linear momentum is conserved: $m_A v_A = m_A v'_A + m_B v'_{CM}$, where v'_{CM} is the velocity of the center of mass of the bar after impact, and v_A, v'_A are the velocities of the sphere before and after impact. The angular momentum about the point of impact is conserved:



$0 = -\left(\frac{L}{2}\right) m_B v'_{CM} + I_{CM}\omega'$. From Eq (8.46) (see solution to Problem 8.62) if the sphere adheres to the bar, $v'_A = v'_{tip}$, and $e = 0$. From kinematics,

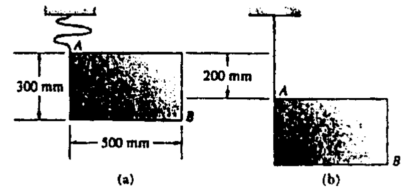
$v'_{tip} = v'_{CM} + \left(\frac{L}{2}\right)\omega'$, from which $v'_{CM} = v'_A - \left(\frac{L}{2}\right)\omega'$. Substitute into the expressions for the conservation of momentum to obtain the two equations in two unknowns: $m_A v_A = (m_A + m_B)v'_A - \left(\frac{L}{2}\right) m_B \omega'$, and



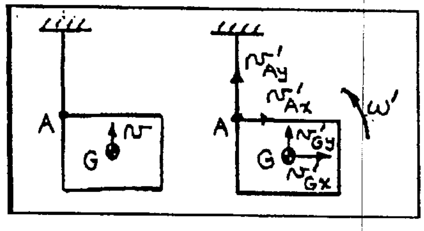
$0 = -\left(\frac{L}{2}\right) m_B v'_A + \left(\frac{m_B L^2}{4} + I_{CM}\right)\omega'$. Note $I_{CM} = \frac{m_B L^2}{12}$. Solve:

$$\boxed{\omega' = \frac{6 m_A v_A}{(4m_A + m_B)L} = 3.75 \ rad/s}$$

6 The 20 *kg* homogenous rectangular plate is released from rest (Fig a) and falls 200 *mm* before coming to the end of the string attached to the corner A (Fig b). Assuming that the vertical component of the velocity of A is zero just after the plate reaches the end of the string, determine the angular velocity of the plate and the magnitude of the velocity of the corner B at that instant.



Solution Choose a coordinate system with the x axis parallel to top edge of the plate in Fig (a), with the y axis positive upward.



Denote the vector distance from the corner A to the center of mass of the plate by $\vec{r} = a\vec{i} - b\vec{j}$, where $a = 0.25\ m$, $b = 0.15\ m$. From the principle of work and energy, $U = T_2 - T_1$, where $T_1 = 0$ since the plate is released from rest. The work done is $U = \int_0^h -mgdh = -mgh$, where $h = -0.2\ m$. The kinetic energy is $T_2 = \left(\frac{1}{2}\right)mv^2$, from which $v = -\sqrt{-2gh} = -1.981\ m$, where the negative sign on the square root is chosen to conform to the choice of coordinates. From the conservation of linear momentum in the horizontal direction, $mv_{Gx} = mv'_{Gx}$, from which $v_{Gx} = v'_{Gx} = 0$, that is, *the horizontal component of the velocity of the center of mass is zero the instant after the string tightens.* From kinematics, the velocity of the center of mass is $\vec{v}'_G = \vec{v}'_A + \vec{\omega}' \times \vec{r}$.

$$\vec{v}'_G = \vec{v}'_A + \vec{\omega}' \times \vec{r} = \vec{i}v'_{Ax} + \vec{j}v'_{Ay} + \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega' \\ a & -b & 0 \end{bmatrix} = \vec{i}(v'_{Ax} + b\omega') + \vec{j}(v'_{Ay} + a\omega').$$

Since $v'_{Ay} = 0$, $v'_{Gx} = 0$, then $0 = v'_{Ax} + b\omega'$, and $v'_{Gy} = a\omega'$. The angular momentum about A is conserved:

$(\vec{r} \times m\vec{v}_G) = (\vec{r} \times m\vec{v}'_G) + I_G\vec{\omega}'$. Substitute and reduce: $\vec{r} \times m\vec{v}_G = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & -b & 0 \\ 0 & mv & 0 \end{bmatrix} = (amv)\vec{k}$,

$\vec{r} \times m\vec{v}'_G = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & -b & 0 \\ 0 & ma\omega' & 0 \end{bmatrix} = a^2m\omega'\vec{k}$, $I_G\vec{\omega}' = \left(\frac{(2a)^2 + (2b)^2}{12}\right)m\omega'\vec{k}$. Collect terms and substitute

into the conservation of angular momentum expression to obtain $amv = \left[a^2m + \left(\frac{(2a)^2 + (2b)^2}{12}\right)m\right]\omega'$,

from which $\boxed{\omega' = \frac{3av}{(4a^2 + b^2)} = -\frac{3a\sqrt{-2gh}}{(4a^2 + b^2)} = -5.452\ rad/s}$, (clockwise),

$\vec{v}'_G = +(a\omega')\vec{j} = -1.363\vec{j}\ (m/s)$, and $v'_{Ax} = -b\omega' = 0.818\ m/s$. The velocity of the corner B is

obtained from $\vec{v}'_B = \vec{v}'_G + \vec{\omega}' \times (a\vec{i} - b\vec{j}) = 0 + \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega' \\ a & -b & 0 \end{bmatrix} = (b\omega')\vec{i} + (v'_{Gy} + a\omega')\vec{j}$,

$\vec{v}_B = -0.818\vec{i} - 2.73\vec{j}\ (m/s)$, from which $\boxed{|\vec{v}_B| = 2.85\ m/s}$