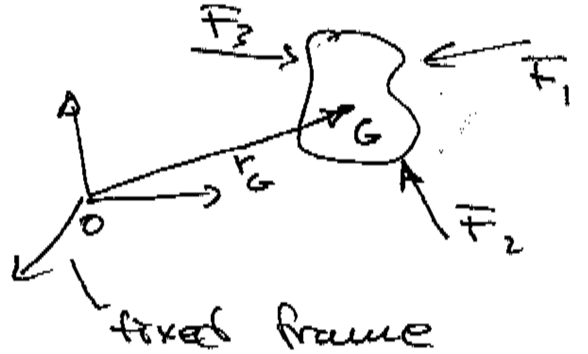
Planar Dynamics of Rigid Bodies

Consider a rigid body



Newton's 2nd Law (fixed mass)

$$\boxed{\Sigma \underset{\sim}{F} = m \cdot \underset{\sim}{a}_G}$$

a_G is the acceleration of the center of mass as measured from the inertial reference frame.

also

$$\Sigma M_o = \frac{d}{dt}(H_o)$$

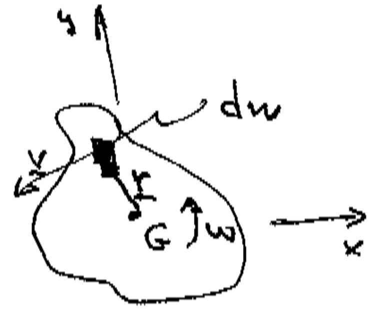
but $\quad H_o = r_G \times m \underset{\sim}{V}_G + H_G$

where H_G is the angular momentum about center of mass and V_G is the velocity of the center of mass.

so we have

$$\Sigma M_o = r_G \times m a_G + \frac{d}{dt}(H_G)$$

What is H_G?



$$\Delta H_G = \underset{\sim}{r} \times dm \underset{\sim}{v} = \underset{\sim}{r} \times dm(\underset{\sim}{\omega} \times \underset{\sim}{r})$$

For planar motion:

$$\Delta H_G = r^2 \omega \, dm \, \underset{\sim}{k} = r^2 \omega \, \rho \, dA \, \underset{\sim}{k}$$

Now integrate

$$H_G = \omega \int_A \rho r^2 dA \, \underset{\sim}{k} = \omega I_G \underset{\sim}{k}$$

where I_G is the moment of inertia about an axis through the center of mass $\perp$ to the plane of motion.

so $\boxed{H_G = \omega I_G \cdot \underset{\sim}{k}}$

Special Cases (important)

1) when O is the center of mass

$$H_O = H_G = I_G \cdot \omega \cdot \underset{\sim}{k}$$

2) when O is on a fixed axis of rotation:

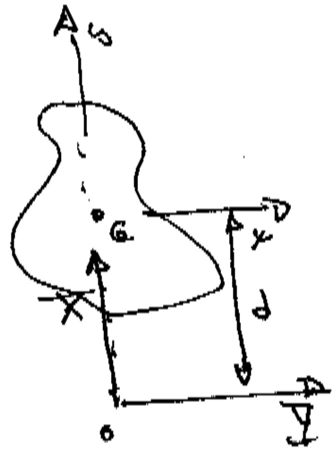
$$H_O = I_O \cdot \omega \cdot \underset{\sim}{k}$$

since $H_O = r_G \times m V_G + I_G \cdot \omega$

$$= r_G \times m v_G + (I_O - m r^2)\, \omega\, \underset{\sim}{k}$$

$$H_O = I_O \cdot \omega\, \underset{\sim}{k}$$

Note fact $\quad I_O = I_{cm} + m d^2$



Summary:

For planar motion:

$$\sum \underset{\sim}{F}_x = m\, \underset{\sim}{a}_{Gx} \qquad \left(\text{or} \quad \sum F_n = m\, a_{Gn} \right.$$

$$\sum \underset{\sim}{F}_y = m\, \underset{\sim}{a}_{Gy} \qquad \left. \sum F_t = m\, a_{Gt} \right)$$

and

$$\sum M_O = \underset{\sim}{r}_G \times m\, \underset{\sim}{a}_G + \frac{d\underset{\sim}{H}_G}{dt}$$

$$= \underset{\sim}{r}_G \times m\, \underset{\sim}{a}_G + I_G \cdot \underset{\sim}{\omega}$$

Special Cases

① Center of mass

$$\Sigma F_x = m\, a_{Gx}$$

$$\Sigma F_y = m\, a_{Gy}$$

$$\Sigma M_G = I_G \cdot \alpha$$

② Rotation about a fixed point O

$$\Sigma F_x = m\, a_{Gx}$$

$$\Sigma F_y = m\, a_{Gy}$$

$$\Sigma M_O = I_O\, \alpha$$