Sample Problems

Math 464

The test is closed book. One notebook sized page of notes will be allowed on the test. You may use your calculator on problem number 1, but on all other problems, you should give any numerical answers in the form of a rational number or a simple expression involving radicals.

1. A machine uses the decimal system and represents real numbers in the form $\pm .m_1 m_2 m_3 m_4 \times 10^e$ where each m_j is an integer, $0 \le m_j \le 9$, $m_1 \ne 0$, and $-5 \le e \le 5$. The machine rounds to nearest even. On this machine the following formula is used to compute a function f:

$$f(x) = \sqrt{x-1} - \sqrt{x-2}$$

   a) How many digits of accuracy will you get for $f(100)$? What is the relative error?
   b) Suggest an alternate formula which is more accurate. What is the relative error?

2. Each of the following iterations may converge. If the iteration does converge it will be to a root of a polynomial $p(x)$. In all three cases the convergence will be to some root of the same polynomial.

$$x_{k+1} = x_k^2 - 2x_k + 2 \tag{1}$$

$$x_{k+1} = x_k - \frac{x_k^2 - 3x_k + 2}{2x_k - 3} \tag{2}$$

$$x_{k+1} = -x_k^2 + 4x_k - 2 \tag{3}$$

   a) What is $p(x)$? What are its roots?
   b) One of the iterations will converge to either of the two roots of $p(x)$ if the initial guess is sufficiently close. Each of the others will converge to one root, but can't possibly converge to the other unless the initial guess is exactly equal to the root. Which is which? Give a justification for your answer.

3. a) Find the $PA = LU$ factorization of

$$\begin{bmatrix} 1 & 2 & 2 \\ 4 & 8 & 16 \\ -2 & -1 & -2 \end{bmatrix}$$

   by using Gaussian elimination and pivoting. L should have 1's on the main diagonal.

b) Use part a) to solve:

$$Ax = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

4. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 5 \\ 0 \\ -4 \end{bmatrix}$. Find the least squares solution to the problem $Ax = b$ using the normal equations.

5. a) Let A be row diagonally dominant. Prove that A^{-1} exists.
   b) Let

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Compute two steps, $x^{(1)}, x^{(2)}$ of the Jacobi method for solving $Ax = b$. Give an error estimate for $\|x^{(2)} - x_t\|$ in terms of $\|x^{(1)} - x^{(0)}\|$, where x_t is the true solution.

6. Let

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

   a) Compute $\kappa_\infty(A)$.
   b) Let

$$b = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

   and suppose that

$$x_c = \begin{bmatrix} .9 \\ 1.1 \end{bmatrix}$$

   is a computed solution of $Ax = b$. Give upper and lower estimates on $\frac{\|x_c - x_t\|}{\|x_t\|}$, using $\kappa_\infty(A)$, where x_t is the true solution.

7. Consider the fixed point iteration

$$x_{n+1} = \frac{1}{3 + x_n}, \; x_0 = 0$$

   a) Prove that

$$|x_{n+1} - x_n| < \frac{1}{9}|x_n - x_{n-1}|$$

   b) Prove that x_n converges to a limit, s.
   c) Prove that $x_n - s$ alternates in sign.
   d) Find s.