MATH 407 | SUMMER 2001 | July 12, 2001

ABOUT MIDTERM | David L. Ragozin

The Midterm next Thursday, July 19, will cover Chapters 1 to 3 of the course notes. It will contain eight questions (so allow about 7-8 minutes per question). Questions on this exam will be similar in nature to those in Quizzes 1-3, in the lectures, and in homeworks 1 to 3. It is important that you be proficient at

(i) Solving an LP in 1 variable with unknown coefficients,
(ii) Solving graphically an LP in two variables (say, under 5 minutes for one defined by 5 linear equations/inequalities),
(iii) recognizing or creating LP's in the various standard forms,
(iv) performing a feasible pivot (say, under 5 minutes for a system in 6 variables and with 4 equations) and
(v) doing p̀roofsóf basic facts, similar to the justifications given in lectures (some of the questions will require you to prove similar results.)

Some additional questions for practice are given below. Good luck! [Disclaimer: These are ONLY practice questions so they need not bear any resemblance to the exam questions.]

1.

a) What is a linear function? What is a linear program? [Describe in words if possible.]

b) Transform the following problem into a linear program:

$$\begin{array}{ll} \min & 3x_1 + x_2 \\ \text{s.t.} & \dfrac{x_1}{x_1+x_2+x_3} = .5 \\ & \dfrac{x_3}{x_1+x_2+x_3} \leq .7 \\ & x_1 \geq 0,\ x_2 \geq 2, x_3 \geq 2. \end{array}$$

[The above LP models a situation in which three raw materials are mixed to form a product, and the proportion of raw material 1 (respectively, 3) in this product is required to be equal to .5 (respectively, below .7).]

2. Consider the LP:

$$\begin{array}{llcl} \max & c_1x_1 & & \\ \text{s.t.} & a_1x_1 & \geq & 1. \end{array}$$

Here a_1, c_1 are some numbers.

a) For what values of a_1 does the LP have a feasible solution? [Give all possible values.]

b) For what values of a_1 and c_1 does the LP have a unique optimal solution? [Give all possible values.]

c) For what values of a_1 and c_1 is the cost function of the LP unbounded from above on the feasible region? [Give all possible values.]

3. Consider the LP:

$$\begin{array}{lrcrcl} \max & -2x_1 & + & x_2 & & \\ \text{s.t.} & 12x_1 & + & 3x_2 & \leq & 6 \\ & -3x_1 & + & x_2 & \leq & 1 \\ & & & x_2 & \leq & 10 \\ & x_1 & & & \geq & 0. \end{array}$$

a) Graph the feasible region (label all vertices and inequalities). Identify all redundant equations/inequalities, if any.

b) Find an optimal solution and the optimal cost, if there is any. You should check your answer by graphing the set of points whose cost is less than or equal to the optimal cost.

c) Repeat part (b) but with the cost function changed to $5x_1 + x_2$.

4. Consider the LP:

$$\begin{array}{lrcrcl} \max & c_1x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & a_{12}x_2 & = & 1 \\ & x_1 & & & \geq & b_2 \\ & & & x_2 & \geq & 0. \end{array}$$

Here a_{12}, b_2, c_1 are numbers to be specified.

a) Give a value of a_{12}, b_2 and c_1 so that the LP has no feasible solution.

b) Give a value of a_{12}, b_2 and c_1 so that the LP has a unique optimal solution.

c) Give a value of a_{12}, b_2 and c_1 so that the LP has multiple optimal solutions.

d) Give a value of a_{12}, b_2 and c_1 so that the cost function of the LP is unbounded from above on the feasible region.

5. Consider the following system of linear equations:

$$\begin{array}{ccccccccccccc}
x_1 & & & + & 3x_3 & & & - & 1.5x_5 & & & = & 9 \\
 & & & - & 2x_3 & + & x_4 & - & 2x_5 & & & = & 5 \\
 & & x_2 & + & 2x_3 & & & - & 4x_5 & & & = & 4 \\
 & & & - & x_3 & & & + & .5x_5 & + & x_6 & = & -23.
\end{array}$$

Is this system in feasible standard form? If yes, make a feasible pivot on a coefficient in column 3 so to transform this system into another system in feasible standard form. Is it possible to make a feasible pivot on a coefficient in column 5? Explain.

6. Consider the LP:

$$\begin{array}{lcccccccccccc}
\min & & & & & & & & & & x_5 & & \\
\text{s.t.} & & & & 1.5x_2 & + & x_3 & + & a_{14}x_4 & & & = & 2 \\
 & x_1 & + & & x_2 & & & + & a_{24}x_4 & & & = & 1 \\
 & & - & & 2x_2 & & & + & a_{34}x_4 & + & x_5 & = & 3
\end{array}$$

$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0.$$

a) Specify the range of the three numbers a_{14}, a_{24}, a_{34} over which the system of linear equations is in optimal standard form.

b) Specify the range of the three numbers a_{14}, a_{24}, a_{34} over which the system of linear equations is in unbounded standard form.

c) Choose a specific value of a_{14}, a_{24}, a_{34} in the range found in part (b). For the value chosen, find a feasible solution of the LP whose cost is less than or equal to $-M \cdot a_{34}$, where M is any positive number. [Your feasible solution should depend on M. Your value of a_{14}, a_{24}, a_{34} should NOT depend on M.]

7. Suppose that the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, ..., m,$$

is in optimal standard form. Show that any value of $x_1, ..., x_n$ that satisfies this system AND the inequalities

$$x_j \geq 0, \quad j = 1, ..., n-1,$$

must also satisfy $x_n \geq b_m$.

8. Suppose that a_1, a_2, a_3 are three positive numbers and b_1, b_2, b_3 are three nonnegative numbers such that

$$b_1/a_1 \leq b_2/a_2 \leq b_3/a_3.$$

Determine the sign of each of the following three numbers:

$$b_1/a_1, \quad b_2 - a_2 b_1/a_1, \quad b_3 - a_3 b_1/a_1.$$

9. Consider the following LP in feasible standard form:

$$\begin{array}{lccccccccccccc}
\min & & & & & & & & & & & x_6 & & \\
\text{s.t.} & -2x_1 & - & x_2 & & & + & x_4 & & & & & = & 1 \\
 & 4x_1 & + & 2x_2 & & & & & + & x_5 & & & = & 6 \\
 & 2x_1 & + & 4x_2 & + & x_3 & & & & & & & = & 2 \\
 & -x_1 & + & 3x_2 & & & & & & & + & x_6 & = & 0
\end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0.$$

Is the system of linear equations in optimal standard form or in unbounded standard form? If neither, apply ONE simplex iteration to the above LP.

10. Consider the LP in the n variables $x_1, ..., x_n$:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, ..., m, \end{aligned}$$

where a_{ij}, $i = 1, ..., m, j = 1, ..., n$, and b_i, $i = 1, ..., m$, and c_j, $j = 1, ..., n$, are given numbers. Assume that this LP has an optimal solution. Show that (i) the following transformed LP in the $n + m + 1$ variables $x_1, ..., x_n, s_1, ..., s_m, z$:

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j + s_i = b_i, \quad i = 1, ..., m, \\ & \sum_{j=1}^{n} c_j x_j + z = 0, \\ & \quad s_i \leq 0, \quad i = 1, ..., m, \end{aligned}$$

also has an optimal solution and (ii) if $x_1 = x_1^*, ..., x_n = x_n^*, s_1 = s_1^*, ..., s_m = s_m^*, z = z^*$ is any optimal solution of this transformed LP, then $x_1 = x_1^*, ..., x_n = x_n^*$ is an optimal solution of the original LP. [NOTE: This question is long, so such a question will likely not appear in the exam in its entirety.]

Note. Although you will be allowed two sheets of notes, it is important that you do not become reliant on written notes. As you have seen with the quizzes it is important to know the concepts so you can begin dealing with the content of the problem right away.