

A few words about the final, both the take home out on Wednesday and the inclass portion on Friday. First, the take portion will be primarily on Chapter 7 though you can't do this work without understanding Chapters 3-6. Second the in class portion will be comprehensive - Chapters 1 through 7, though anything on Chapter 7 will be brief, and obviously the excluded material in 3.4, 6.5 will not be covered. Questions on the final will be similar in nature to those in earlier exams, quizzes and in lectures, and in homeworks. The exam will be 60 minutes long and will contain about six questions. About 15%-20% of the questions will be of the proof-type. Two 8.5" by 11" sheets of handwritten notes(one side each or one sheet both sides) will be allowed for the final, and you will be able to refer to your solutions to the take-home portion. Some practice questions for Chs. 4, 5, 6 are given below. Working on the take home portion should also provide some preparation for the in-class portion. Good luck!

Exercise 6.2.5

Exercises 6.4.2, 6.4.3

1. In answering the following parts, you must also explain your answer (by invoking either weak duality or strong duality or both).

- What does weak duality say about an LP and its dual? What does strong duality say about an LP and its dual?
- Suppose that the dual of an LP has no feasible solution. What does this say about the original LP?
- Suppose that the dual of an LP does not have an optimal solution. What does this say about the original LP?
- Suppose that the dual of an LP has an optimal solution. What does this say about the original LP?
- Suppose that both an LP and its dual have feasible solutions. What does this say about their respective optimal costs? Do the two LPs have optimal solutions?

2. Consider the following LP:

$$\begin{array}{ll} \max & 2x_1 + x_2 - x_3 \\ \text{s.t.} & -x_1 + 2x_2 + x_3 = 2 \\ & x_1 + x_2 + x_3 = 0 \\ & x_2 \leq 0, x_3 \geq 0. \end{array}$$

- Write down the dual of the given LP.
- Find an optimal solution to the dual LP found in part (a).
- Write down the complementary slackness condition between the original LP and the dual LP.
- Find an optimal solution to the original LP. [Hint: Use your answer to part (c).]

3. Consider the LP

$$\begin{array}{ll} \max & 2x_1 + 2x_2 + 2.5x_3 \\ \text{s.t.} & .2x_1 + .3x_2 + .2x_3 \leq 13 \\ & .3x_1 + .3x_2 + .4x_3 \leq 22 \\ & .5x_1 + .4x_2 + .4x_3 \leq 40 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{array}$$

- Write down the dual of this LP.
- Write down the dual of the dual LP found in part (a). [You should be able to do this with little effort.]
- Following Example 6.2.1 and the examples given in class, show that the given LP and its dual found in part (a) satisfy the weak duality relation. [In other words, you must show that the cost of ANY feasible

solution to the original LP is less than the cost of ANY feasible solution to the dual LP. Your proof should be entirely algebraic.]

- d) After introducing slack variables x_4 , x_5 , and x_6 associated with the three “ \leq ” inequalities and a cost variable x_7 , the above LP is converted into the following LP in feasible standard form:

$$\begin{aligned} \min \quad & x_7 \\ \text{s.t.} \quad & \begin{pmatrix} .2 & .3 & .2 & 1 & 0 & 0 & 0 \\ .3 & .3 & .4 & 0 & 1 & 0 & 0 \\ .5 & .4 & .4 & 0 & 0 & 1 & 0 \\ 2 & 2 & 2.5 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_7 \end{pmatrix} = \begin{pmatrix} 13 \\ 22 \\ 40 \\ 0 \end{pmatrix} \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0. \end{aligned} \tag{1}$$

Upon applying the simplex method to this LP, we find the following system of linear equations equivalent to (1):

$$\begin{pmatrix} 0 & -5 & 0 & -4 & 1 & 1 & 0 \\ 1 & 3 & 0 & 20 & -10 & 0 & 0 \\ 0 & -1.5 & 1 & -15 & 10 & 0 & 0 \\ 0 & -2.5 & 0 & -2.5 & -5 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_7 \end{pmatrix} = \begin{pmatrix} 10 \\ 40 \\ 25 \\ -142.5 \end{pmatrix}, \tag{2}$$

with basis $\{x_6, x_1, x_3, x_7\}$. Find an optimal solution of the original LP (not the transformed LP).

- e) Write down the complementary slackness condition for the original LP and the dual LP found in part (a). Use this condition to find an optimal solution of the dual LP. [You can check that what you find is indeed an optimal solution by checking that it is a feasible solution of the dual LP and its cost equals the optimal cost of the original LP.]

4. Determine whether the following LP has a feasible solution or not.

$$\begin{aligned} \min \quad & -x_1 - 2x_2 + 2x_3 \\ \text{s.t.} \quad & -2x_1 - x_2 + x_3 \geq 1 \\ & x_1 - x_2 - x_3 \geq 0 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

[Hint: Look at its dual.]

5. Consider the LP:

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j = b_i, \quad i = 1, \dots, m, \\ & x_j \geq 0, \quad j = 1, \dots, n. \end{aligned}$$

Show that this LP has no optimal solution whenever the cost function of the following modified LP:

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j z_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} z_j = 0, \quad i = 1, \dots, m, \\ & z_j \geq 0, \quad j = 1, \dots, n. \end{aligned}$$

is unbounded above on the feasible region.

6. Exercise 4.2.6.

7. Let

$$\sum_{j=1}^n a_{ij}x_j = b_i, \quad i = 1, \dots, m,$$

be a system of linear equations in dual-feasible standard form. Suppose that we make a dual-feasible pivot on a row $r \in \{1, \dots, m-1\}$ with $b_r > 0$. What can you say about the m th right-hand coefficient of the new system compared to b_m ? Is it less than or equal to b_m ? Give a proof for your answer.

8. Let

$$\sum_{j=1}^n a_{ij}x_j = b_i, \quad i = 1, \dots, m,$$

be a system of linear equations in dual-feasible standard form. Suppose that after making a dual-feasible pivot on a row $r \in \{1, \dots, m-1\}$ with $b_r \neq 0$, we notice that the m th right-hand coefficient of the new system equals b_m . Show that the (m, j) th left-hand coefficient of the new system equals a_{mj} for all $j = 1, \dots, n$.