Consider the two matrices: $A = \begin{bmatrix} 1/2 & 5/2 \\ 5/2 & 1/2 \end{bmatrix}$ and $G = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ It is easy to construct $G^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and verify the columns of G are orthogonal and

$$H = G^{-1}AG = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \qquad (*).$$

Computations with a diagonal matrix like H are easy, for instance

$$H^{100}\mathbf{x} = \begin{bmatrix} 3^{100}x_1 \\ (-2)^{100}x_2 \end{bmatrix}, \qquad (**)$$

so $H^{100} = \begin{bmatrix} 3^{100} & 0 \\ 0 & (-2)^{100} \end{bmatrix}$ More generally, for any polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n$,

$(a_0 I + a_1 H + \cdots + a_n H^n)\mathbf{x} = p(H)\mathbf{x} = \begin{bmatrix} p(3)x_1 \\ p(-2)x_2 \end{bmatrix}, \qquad (***)$

so $p(H) = \begin{bmatrix} p(3) & 0 \\ 0 & p(-2) \end{bmatrix}$.

In this project you are to **verify** the statements above, **develope** similar relations between other (symmetric) A's, other G's with orthogonal columns and other diagonal H's and **use** the relations to simplify similar computations with A such as computing $A^{100}\mathbf{x}$ or $p(A)\mathbf{x}$. Also you are going to **explore** how the concepts of eigenvalues and eigenvectors from Chapter 4, for (symmetric) A, relate to the diagonal H's and the orthogonal columns of G's which produce relations such as (*).

In a group with 4 other people you should consider the following three investigation topics. Your group will hand in one report which presents your findings/conclusions/solutions with respect to these investigation topics. As usual all dicussions must show the work done to reach your conclusions and also give arguments and reasoning which explain what you have done and why. You may submit a **preliminary** version of your report on Part 1 on Fri May 30 in which case comments on your preliminary version will be returned in class June 2. (Preliminary versions can be submitted on June 2 but then will only be returned on June 4.)

Your groups **final report** which is to include **Part I** (the preliminary commented version should be an attachment if it was done) and **Part II** is due *the last day of class* (but an extension for part II until the day of **final exam** will be granted on request - just note your request on your part I submission).

Part II consists of several problems mostly relating to the last 4 sections of Chapter 4 (4.5-4.8). This part II *should not be part of any preliminary report*.

1.

(a) Verify the orthogonality statements about the columns of G, the formula for G^{-1} and (*).

(b) Show that (**) above is true when 100 is replaced by 2, 5 or more generally any n. Also explain why (***) above is true. Explanations which leave numbers which arise, such as 3^5 or $(-2)^2$, unexpanded should be given, whenever possible.

(c) Now figure out how to use (*), (**) and/or (***) to compute (moderately simple numerical expressions for) A^2, A^{10}, A^{100}. (You'll want to verify and use the fact that $GHG^{-1} = A$ and some simple algebra to express A^n in terms of H^n.

(d) More generally, show that for any polynomial $p(t)$, if $\mathbf{x} = G\begin{bmatrix} a \\ b \end{bmatrix}$, then $p(A)\mathbf{x} = G\begin{bmatrix} p(3)a \\ p(-2)b \end{bmatrix}$.

This provides efficient ways to compute polynomials in A applied to $\mathbf{x}$. Use it to get explicit formulae for three $p_i(A)\mathbf{x}, i = 1, 2, 3,$ where $p_1(t) = 1 + 4t + 4t^2$, $p_2(t) = -6 - t + t^2$, $p_3(t) = 1000 - 5t^8 + t^{49}$. (Expressions like 3^8 are more useful in understanding such formulae than the full expansion, 6561, of such an expression.)

2. The investigations in 1 should convince you that it is (computationally) useful to be able to write a given matrix A in the form $A = GHG^{-1}$ where H is a **diagonal** matrix (notice the shift from (*), which said $H = G^{-1}AG$). In this part you will try to write other (2×2) matrices (other A's) in a similar form.

   (a) First investigate matrices $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ with equal diagonal entries and equal off diagonal entries. For such A can you find G (either the same or different from G above) such that $G^{-1}AG$ is diagonal?

   (b) Second, suppose A is any (2×2) **symmetric** matrix (with non-equal diagonal entries). Can you find an invertible G with orthogonal columns (probably different from the one above) and a diagonal H (almost certainly different from the one above) with $G^{-1}AG = H$? Investigate. See what conditions/equations you can find to help determine the 4 entries of G and the diagonal entries of H.

3. Another way of looking at the fact that $G^{-1}AG = H$ is diagonal, is to write $G = [\mathbf{g}_1 \ \mathbf{g}_2]$ with $\mathbf{g}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{g}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   (a) Show that the equation for H can be rewritten as $AG = GH$ or $[A\mathbf{g}_1 \ A\mathbf{g}_2] = [3\mathbf{g}_1 \ (-2)\mathbf{g}_2]$ using the fact that 3 and -2 were the diagonal entries of H. And explain how this relates the columns of G and the entries of H to eigenvectors and eigenvalues of A.

   (b) Use the invertibility criterion for (2×2) matrices phrased in terms of Δ from §1.10, p.98, to investigate for which numbers λ the matrix $A - \lambda I$ is singular, for a general symmetric $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, as considered in 2(b) above. This might suggest how in the general context of 2(b), when A is an arbitrary (2×2) symmetric matrix, you can find the diagonal entries of the H. If possible, give a formula for the diagonal entries of H.

   (c) Once you have the possible diagonal entries of H associated with an A as in part (b), try to find some orthogonal columns to go into a general G, with $AG = GH$, for the general symmetric A as in (b). Explain what equations you solve to get those columns and why they give an invertible G. If possible show why those columns are orthogonal.

   (d) Work through the details of (b),(c) for the matrices $B = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & -12 \\ -12 & 6 \end{bmatrix}$ in the role of A. The goal is to find G_B with $G_B^{-1}BG_B$ diagonal and to find G_C with $G_C^{-1}CG_C$ diagonal.

   (e) Use the original A, G, H in (*) above. Let $\mathbf{g}_1, \mathbf{g}_2$ be as in the start of part 3, let $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, and let $p(t) = 5t^{10} - 7t^2$. Show that if $\mathbf{x} = a\mathbf{g}_1 + b\mathbf{g}_2$ then $p(A)\mathbf{x} = p(3)a\mathbf{g}_1 + p(-2)b\mathbf{g}_2$.