Supplements to Johnson, Riess and Arnold

David L. Ragozin

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Revised for 5th edition

1 Matrices and Systems of Linear Equations

1.1 Introduction to Matrices and Systems of Linear Equations

Ragozin's quick tricks for elimination. This approach has less arithmetic than the approach in the printed text and allows decisions to be made sooner about the existence of solutions. All the examples and solutions I will give follow this slight variant of Johnson, Riess and Arnold.

Replace the material in JRA p.11, from "The variable x_1 has now been ... " through JRA p.12, "... ***Gauss-Jordan elimination***" with the material on the following two pages:

This space intentionally left blank.

The variable x_1 has now been eliminated from the second and third equations. Next, we focus on the last two equations. Our goal is to have x_2 appear here with coefficient 1 in the second equation and then to eliminate x_2 from the remaining equations. We continue the reduction process with the following operations:

System:	***Augmented Matrix:***
(-1)$\boldsymbol{E}_2$:	**(-1)**$\boldsymbol{R}_1$:
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ 2x_2 + x_3 &= -2 \end{aligned}$$	$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$
$\boldsymbol{E}_3$-$2E_2$:	$\boldsymbol{R}_3$-$2R_2$:
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ -3x_3 &= -6 \end{aligned}$$	$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & -6 \end{bmatrix}$$

The variable x_2 has now been eliminated from the third equation. Next we make x_3 have coefficient 1 in the third equation and back eliminate x_3 from the first and second equations:

System: ***Augmented Matrix:***

(-1/3)$\boldsymbol{E}_3$: **(-1/3)$\boldsymbol{R}_3$**:

$$\begin{aligned} x_1 + 2x_2 - \ \ x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\boldsymbol{E}_2$-2$\boldsymbol{E}_3$: **$\boldsymbol{R}_2$-2$\boldsymbol{R}_3$**:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= \ \ 1 \\ x_2 \qquad &= -2 \\ x_3 &= \ \ 2 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\boldsymbol{E}_1$+$\boldsymbol{E}_3$: **$\boldsymbol{R}_1$+$\boldsymbol{R}_3$**:

$$\begin{aligned} x_1 + 2x_2 \qquad &= \ \ 3 \\ x_2 \qquad &= -2 \\ x_3 &= \ \ 2 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Now all that remains is to back eliminate x_2 from the first equation:

System: ***Augmented Matrix:***

$\boldsymbol{E}_1$-2$\boldsymbol{E}_2$: **$\boldsymbol{R}_1$-2$\boldsymbol{R}_2$**:

$$\begin{aligned} x_1 \qquad\qquad &= \ \ 7 \\ x_2 \qquad &= -2 \\ x_3 &= \ \ 2 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The last sytem above clearly has a unique solution given by $x_1 = 7$,$x_2 = -2$, and $x_3 = 2$. Because the final system is equivalent to the original given system, both systems have the same solution.

The reduction process used in the preceding example is known as ***Gauss elimination and back substitution*** ... I.e., continue with “and will be explained in Section 1.2.”

1.2 Echelon Form and Gauss-Jordan Elimination

The box on page 20 gives steps for forming a reduced echelon form matrix which is row equivalent to a given matrix. When the matrices involved are considered as augmented matrices corresponding to systems of linear equations, the corresponding elementary operations on equations comprise what is often called ***Gauss elimination with backsolving***. The first 5 steps, which produce a matrix in echelon form row equivalent to the original matrix, are the Gauss elimination steps. Step 6, which produces a reduced echelon form matrix row equivalent to the original matrix, corresponds to backsolving.

The paragraph on page 20 just before Example 3, starts by saying Example 3 illustrates the six-step process, but immediately goes on to say it doesn't actually illustrate it. Rather it illustrates a "single-pass variation". What is written certainly does "Use elemntary row operations to transform the following matrix to reduced echelon form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 8 & 4 \\ 0 & 0 & 0 & 1 & 3 & 11 & 9 \\ 0 & 3 & -12 & -3 & -9 & -24 & -33 \\ 0 & -2 & 8 & 1 & 6 & 17 & 21 \end{bmatrix}''$$

Here are **replacements** for the steps in Example 3, pages 20-21 which first use steps 1-5 on the matrix above to obtain a row equivalent echelon form matrix and then carry out step 6 to get a row equivalent reduced echelon form matrix:

$\mathbf{R_1} \leftrightarrow \mathbf{R_3}, \mathbf{(1/3)R_1}$: Introduce a leading 1 into the first row of the first nonzero column

$$\begin{bmatrix} 0 & 1 & -4 & -1 & -3 & -8 & -11 \\ 0 & 0 & 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 0 & 0 & 2 & 8 & 4 \\ 0 & -2 & 8 & 1 & 6 & 17 & 21 \end{bmatrix}$$

$\mathbf{R_4 + 2R_1}$: Introduce 0's below the leading 1 in row 1

$$\begin{bmatrix} 0 & 1 & -4 & -1 & -3 & -8 & -11 \\ 0 & 0 & 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 0 & 0 & 2 & 8 & 4 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 \end{bmatrix}$$

$\mathbf{R_4 + R_2}$: Introduce 0's below the leading 1 in row 2

$$\begin{bmatrix} 0 & 1 & -4 & -1 & -3 & -8 & -11 \\ 0 & 0 & 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 0 & 0 & 2 & 8 & 4 \\ 0 & 0 & 0 & 0 & 3 & 12 & 8 \end{bmatrix}$$

$\mathbf{(1/2)R_3}$: Introduce a leading 1 into row 3

$$\begin{bmatrix} 0 & 1 & -4 & -1 & -3 & -8 & -11 \\ 0 & 0 & 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 3 & 12 & 8 \end{bmatrix}$$

$\mathbf{R_4 - 3R_3}$: Introduce a 0's below the leading 1 in row 3

$$\begin{bmatrix} 0 & 1 & -4 & -1 & -3 & -8 & -11 \\ 0 & 0 & 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\mathbf{(1/2)R_4}$: Introduce a leading 1 into row 4

$$\begin{bmatrix} 0 & 1 & -4 & -1 & -3 & -8 & -11 \\ 0 & 0 & 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{R_1 + 11R_4, R_2 - 9R_4,}$ $\mathbf{R_3 - 2R_4}$: Introduce 0's above the leading 1 into row 4

$$\begin{bmatrix} 0 & 1 & -4 & -1 & -3 & -8 & 0 \\ 0 & 0 & 0 & 1 & 3 & 11 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{R_1 + 3R_3, R_2 - 3R_3}$: Introduce 0's above the leading 1 into row 3

$$\begin{bmatrix} 0 & 1 & -4 & -1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{R_1 + R_2}$: Introduce 0's above the leading 1 into row 2

$$\begin{bmatrix} 0 & 1 & -4 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice this example is entirely about reduced echelon form for matrices. Nothing has been said about any equations. (If the given matrix was the augmented matrix of a system, then we could have concluded that the system was inconsistent just after the $\mathbf{R_4 - 3R_3}$ step above. Do you see why?)

To further understand the (slight) difference between Gauss elimination and Gauss-Jordan elimination, I offer **replacements** for the last two reduction steps in Example 4 on page 23; these are part of finding the solutions to a linear system whose augmented matrix is

$$\begin{bmatrix} 2 & -4 & 3 & -4 & -11 & 28 \\ -1 & 2 & -1 & 2 & 5 & -13 \\ 0 & 0 & -3 & 1 & 6 & -10 \\ 3 & -6 & 10 & -8 & -28 & 61 \end{bmatrix}.$$

Notice that the first row equivalence in JRA is not what step 1 and 2 would suggest you should do. Rather it is a "simple" way to produce a leading 1 in column 1 to use for further elimination. Also note that the notation $\mathbf{R_1} + \mathbf{R_2}$ has R_1 occuring first, so that tells us that row 1 has been changed.

$\mathbf{R_3} + \mathbf{3R_2}, \mathbf{R_4} - \mathbf{4R_2}$: Introduce 0's below the leading 1 in row 2

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -6 & 15 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & -2 & -6 & 8 \end{bmatrix}$$

$\mathbf{R_4} + \mathbf{2R_3}$: Introduce 0's below the leading 1 in row 3

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -6 & 15 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{R_1} + \mathbf{2R_3}$: Introduce 0's above the leading 1 in row 3

$$\begin{bmatrix} 1 & -2 & 2 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{R_1} - \mathbf{2R_2}$: Introduce 0's above the leading 1 in row 2

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix above is the augmented matrix for the system of equations: ...
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1.3 Consistent Systems of Linear Equations

On page 32 and 33 of this section there are solutions to two problems which use variants of the basic 6 step method for finding a matrix in reduced echelon form which is row equivalent to a given matrix. Here are the reductions which strictly apply the 6 step method. We have used a slighly modified format to indicate steps, and in particular have used $\mathbf{R_3'}$ (in rows 2 and 3) to indicate that the *new* row 3 (after the multiplication by 1/8) has been used to carry out these steps.

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 2 & 4 & 3 & 1 & 0 \\ 3 & 6 & 6 & 2 & 0 \end{bmatrix} \quad \begin{array}{c} \Longrightarrow \\ \mathbf{R_2 - 2R_1} \\ \mathbf{R_3 - 3R_1} \end{array} \quad \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 3 & -7 & 0 \end{bmatrix} \quad \begin{array}{c} \\ \Longrightarrow \\ \mathbf{R_3 - 3R_2} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{bmatrix} \quad \begin{array}{c} \mathbf{(1/8)R_3} \\ \mathbf{R_2 + 5R_3'} \\ \mathbf{R_1 - 3R_3'} \end{array} \quad \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{c} \mathbf{R_1 - R_2} \\ \Longrightarrow \\ \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

As a check on your understanding of the elimination process, verify that the middle step $\mathbf{R_2 + 5R_3'}$ is the same as the step $\mathbf{R_2} + (\mathbf{5/8})\mathbf{R_3}$, where now $\mathbf{R_3}$ denotes row 3 of the matrix to the left of the equivalence step. Also find a so $\mathbf{R_1} - a\mathbf{R_3}$ achieves the same equivalence as the bottom step $\mathbf{R_1 - 3R_3'}$.