NAME **SAMPLE**

MATH 308 — SAMPLE FIRST EXAM — April 13, 1990

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Closed book exam, except one 8×11 sheet of notes may be consulted. There are 8 questions, with a total of 100 points. **To** *receive full credit* **you must show all your work and give reasons. You may point to and use the results (work) for any problem in solving (explaining) any later problem.**

Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 2 \\ 0 & 0 & 4 & -3 \end{bmatrix}$ and let $\mathbf{a}_i$ denote the i'th column of A. Each of problems 1-3 concern these objects.

1.(10 pts.) If A is the **coefficient** matrix of a linear system, how many *unknowns* does that system have? (Explain.)

Sol. A system with A as coefficient matrix has 4 variables(or unknowns), one for each column of A.

2.(15 pts.) Find a **reduced echelon** matrix which is row equivalent to A.

Sol. Perform the following elementary row ops: $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 2 \\ 0 & 0 & 4 & -3 \end{bmatrix} \overset{===>}{R_2 - 3R_1}$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 4 & -3 \end{bmatrix} \underset{R_3 - 4R_2}{===>} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ R_2 + R_3 \\ ===> \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ ===> \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This last is reduced echelon as its echelon (1'st non-zero in each row to right of first non-zero in preceeding row) and each column with a leading non-zero has all other entries 0.

3.(10 pts.) If $B = [\,\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\,]$, how many solutions does the **homogeneous** matrix equation $B\mathbf{x} = \theta$ have? (Why?) (You may, of course, refer to your solution to #2 if that can help explain.)

Sol. The work in #2, applied to just the first 3 columns (B) plus a column of zero's (for the RHS of $B\mathbf{x} = \theta$) shows: $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 6 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \begin{matrix} \textit{various} \\ \textit{row ops} \\ ===> \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus $B\mathbf{x} = \theta$ has **infinitely many** solutions, as there are 2 non-zero rows in a row equivalent echelon matrix, and thus $3 - 2 = 1$ **unconstrained** variables in any solution. (Alternately, from the work above we read off the solutions: $\mathbf{x} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix}$ with the infinitely many (arbitray) values for x_2 leading to infinitely many solutions.)

4.(15 pts.) Suppose a square matrix J satisfies $J^2 = 4J$, but $J \neq 4I$. Explain why J **must be singular**.

Sol. If J is non-singular, then J^{-1} exists. Multiply through by J^{-1} (on the right) to get: $J^2 J^{-1} = 4JJ^{-1}$. Or $J = JI = JJJ^{-1} = 4JJ^{-1} = 4I$. This contradicts $J \neq 4I$. Hence J^{-1} does not exist and J must be singular.

5.(10 pts.) Suppose the (100×2) matrix $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ satisfies $V\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \theta$.

Explain why { $\mathbf{u}_1$, $\mathbf{u}_2$ } is **dependent**, when $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{u}_2 = -\mathbf{v}_1 + \mathbf{v}_2$.

Sol. Since $V\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ says $2\mathbf{v}_1 - \mathbf{v}_2 = \theta$, we have $\mathbf{v}_2 = 2\mathbf{v}_1$. Plug into the definitions of $\mathbf{u}_i$ to get: $\mathbf{u}_1 = \mathbf{v}^1 + 2\mathbf{v}_1 = 3\mathbf{v}_1$ and $\mathbf{u}_2 = -\mathbf{v}_1 + 2\mathbf{v}_1 = \mathbf{v}_1$. This says $\mathbf{u}_1 = 3\mathbf{u}_2$. Thus the $\mathbf{u}_i$'s form a linearly dependent set since one is a multiple of the other. (Or note: $\mathbf{u}_1 - 3\mathbf{u}_2 = \theta$. Thus $x_1 = 1$, $x_2 = -3$ give a non-zero solution to $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \theta$. And so $\{\mathbf{us}_1, \mathbf{u}_2\}$ is a linearly **dependent** set.) (Or you could attempt to find non-zero $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, with $[\mathbf{u}_1 \;\; \mathbf{u}_2]\mathbf{x} = \theta$. Plugging in the given information leads to the equations $(x_1 - x_2)\mathbf{v}_1 + (x_1 + x_2)\mathbf{v}_2 = \theta$. But this will be satisfied provided $x_1 - x_2 = 2$ and $x_1 + x_2 = -1$, from the equation for V. These two can be solved to get the non-zero solution $x_1 = 1/2$, $x_2 = -3/2$).

6.(15 pts.) Let $F = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$. Find F^{-1}, the inverse of F.

Sol. Apply row operations to $[F \;\; I]]$ to get to $[I \;\; B]$, since then the columns of B solve $F\mathbf{b}_j = \mathbf{e}_j$. Hence you can read off $B = F^{-1} = \begin{bmatrix} -1 & 2 & -4 \\ 1 & -1 & 1 \\ 0 & 0 & 1/2 \end{bmatrix}$. (One sequence of row ops which does the job is $R_2 ===> R_2 - R_1$ to get an echelon matrix. Then $R_2 ===> R_2 - R_3$, $R_1 ===> R_1 - 2R_3$, $R_1 ===> R_1 + 2R_2$, to get a reduced echelon matrix ("back elimination"). And then $R_2 ===> -R_2$, $R_3 ===> (1/2)R_3$ to get ones on the diagonal.

7.(10 pts.) Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, and $N = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$.

a) Which product MN or NM is defined? (Explain.)

Sol. NM is defined since 2 = *number of columns* of the left factor (N) = *number of rows* of the right factor (M) = 2.

b) Calculate the entry in the 2^{nd} row and 1^{st} column of the product defined in a).

Sol. (2,1) entry in NM is just the product of row 2 of N with column 1 of M: $[9\ 10]\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 9 \cdot 1 + 10 \cdot 4 = 49$

8.(15 pts.) Suppose a matrix C is changed by the following elementary row operations, $R_2 - 2R_1$, $R_3 + R_1$, $R_3 + 3R_2$ into the matrix F as represented in the following diagram:

$$C \overset{R_2-2R_1}{====>} D \overset{R_3+R_1}{====>} E \overset{R_3+3R_2}{====>} F = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find C and find all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ with $C\mathbf{x} = \theta$.

Sol. Get back to C by undoing the row operations starting from the right (F) and working backwards ($R_i - aR_j$ undoes $R_i + aR_j$ as operations on R_i:

$$F = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_3 - 3R_2}{===>} E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \underset{R_3 - R_1}{===>} D = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ -1 & -5 & -2 \end{bmatrix} \underset{R_2 + 2R_1}{===>} C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ -1 & -5 & -2 \end{bmatrix}.$$

Solve for x

Since row operations will take $[C\ \theta]$ to $[F\ \theta]$, we can read off the fact that there will be no unconstrained (free) variables in the solution $\mathbf{x}$, as there is no column in the coefficient part of $[F\ \theta]$ without a leading non-zero entry. Hence there are only the trivial solution to $F\mathbf{x} = \theta$ and thus only trivial solution to $C\mathbf{x} = \theta$.

Alternate

Or from the system determined by $[F\ \theta]$, backsolving from the bottom up, we read off the solution: $x_3 = 0$, $x_2 = 0$, and $x_1 = -2x_2 - 3x_3 = -2 \cdot 0 - 3 \cdot 0 = 0$. Thus $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the **only** solution.