Scores by question							Total

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MATH 308 **SAMPLE Second Exam** February 26, 1992

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Closed book exam, except one 8×11 sheet of notes may be consulted. There are 7 questions, each worth 15 points. To *receive full credit* you must show all your work and give reasons. You may use your answer to any question to help answer any later question, if it is relevant.

Here is a sequence of row operations which change a matrix A into a reduced row echelon form.

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 & 7 \\ 2 & 4 & 5 & 6 & 8 \\ 3 & 6 & 6 & 9 & 9 \end{bmatrix} \begin{matrix} ====> \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 4 & 3 & 7 \\ 0 & 0 & -3 & 0 & -6 \\ 0 & 0 & -6 & 0 & -12 \end{bmatrix} \begin{matrix} \\ ====> \\ R_3 - 2R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 4 & 3 & 7 \\ 0 & 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} ====> \\ \frac{-1}{3} R_2 \\ \end{matrix} \begin{bmatrix} 1 & 2 & 4 & 3 & 7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 4R_2 \\ ====> \\ \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Use A and the information above to help answer questions 1-4.

1. Compute **nullity**(A). (As always your written work must "justify" your answer.)

Sol. The nullity of A, $\nu(A)$, is the dimension of the nullspace of A. Since the dimension of the nullspace of A is the number of vectors in a basis for the nullspace, and there will be one such vector for each free variable in the solutions to $A\mathbf{x} = \theta$, we get $\boxed{\nu(A) = 3}$ as x_2, x_4 and x_5 will be free, due to lack of pivots in columns 2,4 and 5 of the echelon form for A.

Alt. From the echelon form we get that the nullspace of A (solutions to $A\mathbf{x} = \theta$) consists of $\mathbf{x}$ which solve

$$\begin{matrix} x_1 & +2x_2 & & +3x_4 & -1x_5 & = 0 \\ & & x_3 & & +2x_5 & = 0 \end{matrix}. \text{ Thus } \mathbf{x} = \begin{bmatrix} -2x_2 - 3x_4 + x_5 \\ x_2 \\ -2x_5 \\ x_4 \\ x_5 \end{bmatrix}.$$

From the basis for the nullspace you get from this by setting, in turn, one of the 3 free variables to 1 and the others to 0, there are 3 vectors in such a basis. Hence, $\boxed{\nu(A) = 3}$ since the basis for nullspace has 3 vectors.

2. Is $\bar{\mathbf{v}}$ = [3 6 2 9 -1] in the **row space** of A? Why?

Sol. The non-zero rows in the echelon form above are a basis of the row space of A, so its a question of whether there is a solution to c_1[1 2 0 3 -1]+c_2[0 0 1 0 2] = [3 6 2 9 0-1]. From the first entry we get $c_1 = 3$ and from the third $c_2 = 2$. Now check if entries 2,4,5 are consistent: $c_1 \cdot 2 = 3 \cdot 2 = 6$, so entry 2 is OK; $c_1 \cdot 3 + c_2 \cdot 0 = 3 \cdot 3 = 9$, so 4'th OK, and $c_1 \cdot -1 + c_2 \cdot 2 = 3 \cdot -1 + 2 \cdot 2 = -3 + 4 = 1 \neq -1$, so NOT ok in 5'th spot. Thus $\boxed{\bar{\mathbf{v}} \text{ is } \textbf{not} \text{ in row space of } A}$

Here's the information on A repeated from page 1.

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 & 7 \\ 2 & 4 & 5 & 6 & 8 \\ 3 & 6 & 6 & 9 & 9 \end{bmatrix} \underset{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 4 & 3 & 7 \\ 0 & 0 & -3 & 0 & -6 \\ 0 & 0 & -6 & 0 & -12 \end{bmatrix} \underset{R_3 - 2R_2}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 4 & 3 & 7 \\ 0 & 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{\frac{-1}{3} R_2}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 4 & 3 & 7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_1 - 4R_2}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Either write $A_5 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ as a linear combination of the other columns of A or explain why that is impossible. The question is equivalent to "Can we find a solution to $x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + x_3\mathbf{A}_3 + x_4\mathbf{A}_4 = \mathbf{A}_5$?" Look at the given A as the augmented matrix for this system of equations. Then the work above shows these equations are consistent, and have solutions $x_1 = -1 - 2x_2 - 3x_4$, $x_3 = 2$, with x_2 and x_4 arbitrary. If we let the free variables be zero we get $\boxed{-\mathbf{A}_1 + 2\mathbf{A}_3 = \mathbf{A}_5}$

4. Find an **orthogonal basis** for $R(A)$, the *range* of A. From the row operations shown we see that $\{\mathbf{A}_1, \mathbf{A}_3\}$ the set of columns of A corresponding to the pivots in the echelon form form a basis for the range of A. To get an orthogonal basis, use Gram-Schmidt (or projections):

Let $\mathbf{u}_1 = \mathbf{A}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Now to get another vector in the range orthogonal to $\mathbf{u}_1$ just take away the projection of $\mathbf{A}_2$ on $\mathbf{u}_1$ from $\mathbf{A}_2$, i.e. let $\mathbf{u}_2 = \mathbf{A}_2 - (\mathbf{u}_1^T\mathbf{A}_2/\mathbf{u}_1^T\mathbf{u}_1)\mathbf{u}_1$, i.e. $\mathbf{u}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{[1\ 2\ 3][4\ 5\ 6]^T}{[1\ 2\ 3][1\ 2\ 3]^T} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6}{1^2 + 2^2 + 3^2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Thus,

$$\boxed{\{\mathbf{u}_1 = [1\ \ 2\ \ 3]^T,\ \mathbf{u}_2 = [12/7\ \ 3/7\ \ -6/7]^T\}}$$

5. In both parts of this question, $T: \mathbf{R}^2 \to \mathbf{R}^2$ is a linear map with $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$, $T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}-2\\3\end{bmatrix}$.

a) Find $T(\bar{\mathbf{e}}_1)$.

Sol. First solve $x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\-1\end{bmatrix} = \mathbf{e}_1$ and find $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{2}$. (Found by reducing matrices or observation and checking). Then by linearity

$$T(\mathbf{e}_1) = \tfrac{1}{2}\,T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + \tfrac{1}{2}\,T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \tfrac{1}{2}\begin{bmatrix}2\\3\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

b) Find the matrix for T, i.e. find a matrix B with $T(\bar{\mathbf{x}}) = B\bar{\mathbf{x}}$ for all $\bar{\mathbf{x}}$ in $\mathbf{R}^2$.

Sol. Since $B = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)]$ the first column of B is $T(\mathbf{e}_1) = \begin{bmatrix}0\\3\end{bmatrix}$ from the previous part. We compute the second column $T(\mathbf{e}_2)$ either by repeating part a) for $\mathbf{e}_2$ or by expressing $\mathbf{e}_2$ as a linear combination of two other vectors for which T is is known. One way to do this is

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix}1\\1\end{bmatrix} - \mathbf{e}_1\right) = \begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}.$$

Hence

$$\boxed{B = \begin{bmatrix}0 & 2\\3 & 0\end{bmatrix}}$$

Alt If $B = \begin{bmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{bmatrix}$, then the given information says $B\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}2 & -2\\3 & 3\end{bmatrix}$. If we take transposes we get a "super" augmented matrix for a system of equations to solve to find B^T:

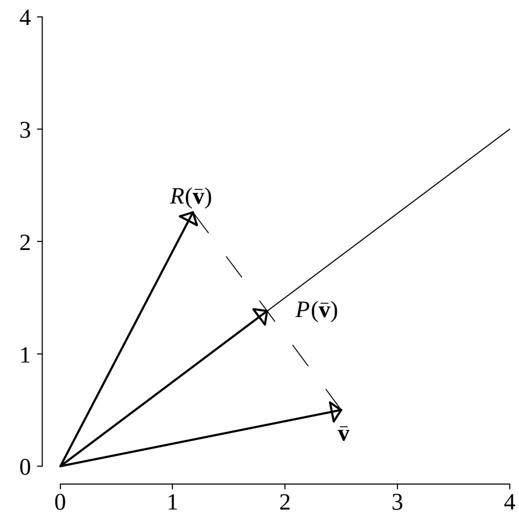
$$\begin{bmatrix}1 & 1 & 2 & 3\\1 & -1 & -2 & 3\end{bmatrix} \overset{\textit{various row ops}}{\Longrightarrow} \begin{bmatrix}1 & 0 & 0 & 3\\0 & 1 & 2 & 0\end{bmatrix}$$

The associated equations say $IB^T = \begin{bmatrix}0 & 3\\2 & 0\end{bmatrix}$ and hence give the B matrix boxed in previous solution.

6. Suppose C is a (10×6) matrix with nullity(C) = 6. Can $C\bar{\mathbf{x}} = \bar{\mathbf{b}}$ be solved for *all* $\bar{\mathbf{b}}$. (As always - explain your answer.)

Sol. Since rank(C) + nullity(C) = number of columns of C, or rank(C) + 6 = 6, the rank(C) = 0. This says the range of C has dimension 0, i.e. the only vector in the range of C is θ. Hence you can only solve $C\mathbf{x} = \mathbf{b}$ when $\mathbf{b} = \theta$, and so CAN NOT solve for all $\mathbf{b}$ in $\mathbf{R}^{10}$.

7. Let R:$\mathbf{R}^2 \rightarrow \mathbf{R}^2$ be symmetric or perpendicular **reflection** in the line l through θ and $\bar{\mathbf{u}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. (For any $\bar{\mathbf{v}}$, its' image, R($\bar{\mathbf{v}}$) is such that P($\bar{\mathbf{v}}$) is the mid-point of the line segment from $\bar{\mathbf{v}}$ to R($\bar{\mathbf{v}}$), where P is the perpendicular *projection* onto l.)



a) Use the mid-point property to find a,b with R($\bar{\mathbf{v}}$) = aP($\bar{\mathbf{v}}$) + $b\bar{\mathbf{v}}$. The fact that $P(\mathbf{v})$ is the mid-point of segment from $\mathbf{v}$ $R(\mathbf{v})$ says $P(\mathbf{v}) = \mathbf{v} + \frac{1}{2}(R(\mathbf{v}) - \mathbf{v})$. Multiply by 2 and solve for $R(\mathbf{v})$ to get $R(\mathbf{v}) = 2P(\mathbf{v}) - \mathbf{v}$. Hence $\boxed{a = 2,\ b = -1.}$

b) Compute R($\begin{bmatrix} 3 \\ 0 \end{bmatrix}$). (You should use the formula for P($\bar{\mathbf{v}}$) involving dot-products and part 7a.) Since $P(\bar{\mathbf{v}}) = (\bar{\mathbf{u}}^T\bar{\mathbf{v}}/\bar{\mathbf{u}}^T\bar{\mathbf{u}})\bar{\mathbf{u}}$, we compute, using 7a,

$$R\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) = 2\,\frac{[4\ 3]\begin{bmatrix} 3 \\ 0 \end{bmatrix}}{[4\ 3]\begin{bmatrix} 4 \\ 3 \end{bmatrix}}\begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 2 \cdot \tfrac{12}{25}\begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 96/25 \\ 72/25 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} =$$

$$\boxed{\begin{bmatrix} 21/25 \\ 72/25 \end{bmatrix}}$$