Energetics of Shallow Water (SW) Waves

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The KE and PE per unit volume are given by

$$KE_V = \frac{1}{2}\rho \mathbf{u}\cdot\mathbf{u}\text{ , and } PE_V = \rho g z$$

Since properties are independent of depth in shallow water flow it is useful (particularly for PE) to consider the depth-integrated energies. This could be considered the first step toward a full volume integral. Taking the depth integrals, and making approximations consistent with the SW equations, we find

$$\frac{KE}{\text{unit horiz. area}} = \int_{-H}^{\eta} \frac{1}{2}\rho \mathbf{u}\cdot\mathbf{u}\, dz \cong \int_{-H}^{0} \frac{1}{2}\rho u^2\, dz = \frac{1}{2}\rho u^2 H = KE_A^{SW}$$
$$\frac{PE}{\text{unit horiz. area}} = \int_{-H}^{\eta} \rho g z\, dz = \frac{1}{2}\rho g\left(\eta^2 - H^2\right) = PE_A^{SW}$$

We can define a zero PE state $PE_{A0}^{SW} = -\frac{1}{2}\rho g H^2$ which is the PE per unit are when the fluid is at rest. It is then customary to call the variation of the PE away from this the "Available PE" or APE. This implies that it is available to be turned back into KE. So we are left with the important quantity

$$APE_A^{SW} = PE_A^{SW} - PE_{A0}^{SW} = \frac{1}{2}\rho g \eta^2$$

note that APE is stored by *any* surface height anomaly, positive or negative (both will give rise to fluid motion, converting APE to KE).

An important property of SW waves (when they are not affected by Earth's rotation) is that they have equal KE and PE. For progressive SW waves with fields given by

$$u = u_0 \cos\left(kx - \omega t\right)$$
$$\eta = \eta_0 \cos\left(kx - \omega t\right)$$

where $\omega/k = c = \sqrt{gH}$ and $u_0 = c\left(\eta_0/H\right)$

then the ratio of the amplitudes of the energies is given by

$$\frac{\left[KE_{A0}^{SW}\right]}{\left[APE_{A0}^{SW}\right]}=\frac{\frac{1}{2}\rho\left(u_0\right)^2 H}{\frac{1}{2}\rho g\left(\eta_0\right)^2}=\frac{c^2\left(\eta_0/H\right)^2 H}{g\left(\eta_0\right)^2}=\frac{gH\left(\eta_0/H\right)^2 H}{g\left(\eta_0\right)^2}=1$$

The property of "equipartition of energy" is true of some other waves, like the deep water waves we will study later in the term. But for SW waves affected by Earth's rotation, called Poincare waves, the ratio becomes greater than 1, meaning the energy is mostly KE. This is particularly true of the near-inertial motions often observed in the ocean. On the other hand, for the quasi-geostrophic flows typical of large scale atmospheric and ocean flows the energy is dominantly stored in the PE.

So, we would like to come up with equations governing the evolution of KE and APE in the SW equations, and in particular for SW waves. One way of doing this is to go back to the full KE and PE equations and write each term consistent with the SW approximations. However, and easier way to get the same result is to form appropriate equations starting from the SW equations themselves. Here's what you do:

First form $\rho g\eta\left(\eta_t+Hu_x=0\right)$

which gives $\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho g\eta^2\right)=-\rho gH\eta u_x}$ $\boxed{APE_A^{SW}}$

also form $\rho Hu\left(u_t=-g\eta_x\right)$

to find $\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 H\right)=-\rho gHu\eta_x}$ $\boxed{KE_A^{SW}}$

The most informative way to look at these is to add them together, creating a single equation for total energy per unit area, which looks like

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 H+\frac{1}{2}\rho g\eta^2\right)=-H\left(\rho g\eta u\right)_x}\quad\boxed{E_A^{SW}}\left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

The term on the left is self-explanatory. To understand the term on the right, let's try taking the integral of the equation over some horizontal area - meaning that we have taken a complete volume integral. Assume that the volume is a long channel of y-direction width B and total length L. At the left end (x=0) assume that there is an open boundary with SW waves propagating into the channel. Also assume that there is some dissipation occurring in the volume that gets rid of the wave energy before it can reflect off the (closed) right end. Then the volume-integrated energy budget would look like:

$$\frac{\partial}{\partial t}\int_{A_0}\left(\frac{1}{2}\rho u^2 H+\frac{1}{2}\rho g\eta^2\right)dA=\left(HB\rho g\eta\right)\times u\Big|_{x=0}-\rho\int_V \varepsilon\, dV$$

where A_0 is the horizontal flat plate area of the volume.

Now we can figure out the physical meaning of the first term on the RHS. It is the "pressure work," expressed here as the area of the open end (*HB*) times the pressure anomaly on that end $(\rho g\eta)$, which gives a force, times the fluid velocity *u* through that end. For our progressive SW waves we can easily calculate this pressure work term. Typically we are concerned with the time average of this term over many wave periods. Let's denote such a "wave average" by angle brackets. Also note the useful integral:

$$\left\langle \cos^2(\omega t) \right\rangle \equiv \frac{1}{2\pi/\omega} \int_t^{t+2\pi/\omega} \cos^2(\omega t)\, dt = \frac{1}{2}$$

So the time-averaged pressure work term is given by

$$\begin{aligned} \left\langle HB\rho g\eta u \right\rangle_{x=0} &= \frac{1}{2} HB\rho g\eta_0 u_0 = \frac{1}{2} HB\rho g\eta_0 c\frac{\eta_0}{H} \\ &= B\left[\frac{1}{2}\rho g\left(\eta_0\right)^2\right]c \end{aligned}$$

The term in square brackets is equal to the wave-averaged total (KE+APE) energy per unit area. This is an example of a rather general property of waves: their energy flux is given by the average energy density times the wave speed *c*. In our case we have already integrated over depth and width, so it is a net flux.

NOTE: in the more general form of this result we have to use the "group velocity" which is defined as $\mathbf{c}_g = \left(c_g^x, c_g^y, c_g^z\right) = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right)$. It happens that for SW waves the group velocity c_g^x is identical to the "phase speed" $\omega/k = c_p^x = \sqrt{gH}$. The group velocity is the speed at which energy travels in waves. We'll study group velocity again when we get to deep water waves.

The key points to remember from this are that:

- The wave properties (like the position of a wave crest, or the energy) travel a lot faster than the fluid velocity of the wave.
- The physical mechanism by which wave energy travels through the fluid is by the pressure work term. This allows the fluid to do work on neighboring parcels, moving energy around, without having to actually *advect* the energy around.