FLUIDS 2009

Problem Set #4 Solutions 11/16/2009

1. Integrate the kinetic energy equation (here given as kinetic energy per unit volume)

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\bullet\mathbf{u}\right)}_{1}=\underbrace{-\nabla\bullet\left[\mathbf{u}\bullet\left(\frac{1}{2}\rho\mathbf{u}\bullet\mathbf{u}\right)\right]}_{2}\underbrace{-\nabla\bullet(\mathbf{u}p)}_{3}+\underbrace{\nabla\bullet\left[\nu\nabla\left(\frac{1}{2}\rho\mathbf{u}\bullet\mathbf{u}\right)\right]}_{4}+\underbrace{p(\nabla\bullet\mathbf{u})}_{5}\underbrace{-\rho g w}_{6}\underbrace{-\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}_{7}$$

over a rectangular volume that goes from the bottom plate to the top plate, for Poiseuille flow

%%% Define the volume of integration, V, to have length L in the x-direction, and width B in the y-direction. Hence $V=2bLB$. The volume integration is quite straightforward, so I won't go into great detail.

Term 1 is zero because the flow is steady.

Term 2 is zero because the inflow of KE_V is balanced by the outflow of KE_V.

Term 3 is readily calculated. Note that the pressure is not a function of y or z (assume gravity = 0). If we define the pressure to be p_1 at the left end of the volume, then it will be $p_1+L\frac{\partial p}{\partial x}$ at the right end of the volume. Thus:

$$-\int_V \nabla\bullet(\mathbf{u}p)\,dV=-\left(p_1+L\frac{\partial p}{\partial x}\right)\int_{A_2}u\,dA+p_1\int_{A_1}u\,dA=-L\frac{\partial p}{\partial x}B\int_{-b}^{b}u\,dz$$

And note that $\int_{-b}^{b}u\,dz=-\frac{2}{3}\frac{b^3}{\mu}\frac{\partial p}{\partial x}$, and $\bar{u}=\frac{1}{2b}\int_{-b}^{b}u\,dz=-\frac{1}{3}\frac{b^2}{\mu}\frac{\partial p}{\partial x}$. Thus

$$-\int_V \nabla\bullet(\mathbf{u}p)\,dV=-LB\frac{\partial p}{\partial x}\left(-\frac{2}{3}\frac{b^3}{\mu}\frac{\partial p}{\partial x}\right)=-V\bar{u}\frac{\partial p}{\partial x}$$

Term 4 is zero because the velocity is zero on the top and bottom walls.

Term 5 is zero because the flow is incompressible, and term 6 is zero because there is no vertical velocity (and gravity = 0).

Term 7, the volume integrated rate of dissipation, is readily calculated as

$$-\mu\int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}dV=-\mu\int_V\left(\frac{\partial u}{\partial z}\right)^2 dV=-BL\mu\int_{-b}^{b}\left(\frac{\partial u}{\partial z}\right)^2 dy=V\bar{u}\frac{\partial p}{\partial x}$$

Thus the balance is between pressure work (which adds energy) and net dissipation, which removes it.

2. Shift your frame of reference so that you are moving to the right with the average flow speed, $\bar{u}$ (that is the average of u over the distance between the plates). This is

called a “Galilean Transform” and the equations of motion are unchanged by such a transformation. Note however that the velocity you observe, call it u' is given by $u' = u - \bar{u}$, where u is the x-velocity in the original frame of reference. Note also that now the solid boundaries are moving. Repeat your volume integral of the kinetic energy equation. Now which of the 7 terms are important? This is an example of how the “story” told by the energy equation may be different depending on the frame of reference.

%%% Here the logic is mostly identical, but now the net pressure work term will be zero, because in the new frame of reference $\overline{u'} = 0$. But the integral arising from term 4 will give exactly the same value as we got from term 3 in part A. Thus the new balance will be between work done by viscous stress on the moving boundaries (which adds energy), and the net loss of energy to dissipation. Note that the dissipation term is unchanged because it involves only velocity gradients.

3. Considering terms in the energy equation within the fluid:

<u>for the original frame of reference</u>

- The pressure work is greatest at z=0, and goes to zero near the top and bottom edges.
- In contrast, the dissipation is zero at z=0 and becomes a maximum at the edges.
- The viscous redistribution term (4) is what allows this: it fluxes energy from its source near z=0 to its sink near the top and bottom edges.