FLUIDS 2009

Problem Set #3 SOLUTIONS 10/27/2009

1.i. The flow is clearly incompressible because

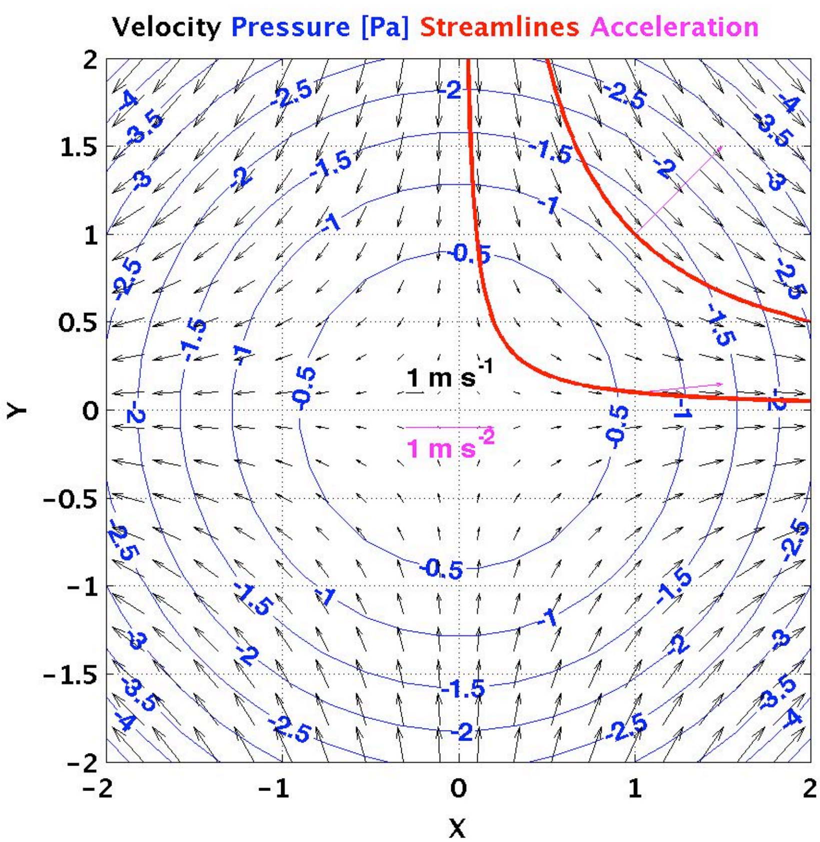
$$\nabla \bullet \mathbf{u} = u_x + v_y + w_z = A - A = 0$$

1.ii. The equation for a streamline is $\frac{dY}{dx} = \frac{-Y}{x}$. Guessing a solution of the form $Y = Cx^n$, this becomes $nCx^{n-1} = -Cx^{n-1}$ which is only satisfied in general if $n = -1$. So the equation of the streamlines is

$$Y = C / x$$

and choosing different values of C places you on different streamlines.

1.iii. Here is a sketch of the flow properties created by the m-file in the Appendix.



The two (red) streamlines have C = 0.1 and C = 1.

1.iv. From the x-momentum equation

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \Rightarrow u u_x = -\frac{1}{\rho} p_x$$

$\Rightarrow p_x = -\rho u u_x = -\rho A^2 x$, and similar reasoning for y-mom gives

$p_y = -\rho A^2 y$

Then integrating these gives the two equations:

$$p = -\frac{1}{2}\rho A^2 x^2 + F(y)$$
$$p = -\frac{1}{2}\rho A^2 y^2 + G(x)$$

then making the obvious choices for the unknown functions *F* and *G*, gives the pressure to within an additive constant:

$$p = -\frac{1}{2}\rho A^2\left(x^2 + y^2\right)$$

So there is relatively high pressure at the origin, and contours of constant pressure are circles, as shown on the diagram above.

1.v. In general the acceleration of a parcel is given by

$$\mathbf{a} = \left(\frac{Du}{Dt}, \frac{Dv}{Dt}\right) = \left(u u_x, v v_y\right) = \left(A^2 x, A^2 y\right)$$

I have drawn acceleration vectors at two locations on the sketch above. On the outer streamline I chose a point where $y = x$. Here the acceleration vector points directly *across* the streamline, indicating that the parcel acceleration is all going into *changing the direction* of parcel motion, and not its *speed*. This is consistent with the pressure gradient at this location, which is normal to the direction of parcel motion.

At the second location, on the inner streamline, I choose a point where $y = x/10$. Here the direction of the acceleration vector is much more closely aligned with the velocity vector, indicating that the acceleration is mainly going into changing the *speed* of the parcel. This is consistent with the direction of motion being "down the pressure gradient."

2.i. Since the solution is a standing wave, assume it has the form given in class:

$$\eta = \eta_0 \cos(kx)\cos(\omega t)$$

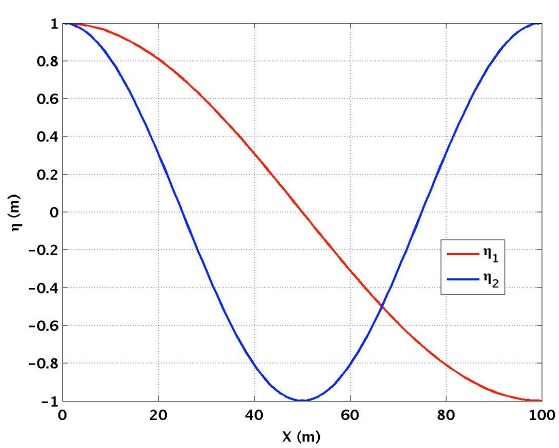
NOTE: you can also have standing waves with other combinations of sines and cosines, depending on the phasing of the two waves used to make it.

The boundary conditions at the two ends of the tanks are that $u(x=0) = u(x=L) = 0$. We showed in class from the x-momentum equation that this requires $\eta_x = 0$ at those two locations, and note that $\eta_x = -k\eta_0 \sin(kx)\cos(\omega t)$. This always satisfies the boundary condition $\eta_x(x=0) = 0$. To satisfy $\eta_x(x=L) = 0$ requires certain values of the wavenumber *k*, specifically $kL = \pi, 2\pi, 3\pi, \ldots$ or $k = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots$ Thus we have figured out the wavenumbers for acceptable solutions. To figure out the frequency of each solution you can use the fact that for any shallow water wave solution the phase speed $c = \frac{\omega}{k} = \sqrt{gH}$, and thus $\omega = k\sqrt{gH} = \frac{\pi}{L}\sqrt{gH}, \frac{2\pi}{L}\sqrt{gH}, \ldots$ The first two of these will clearly be the lowest frequencies. Using subscripts 1 and 2 for these two lowest "modes" we have

Solution #	Wavenumber k	Frequency ω	Period = $2\pi / \omega$
1	π / L	$\pi\sqrt{gH} / L$	20.2 sec.
2	$2\pi / L$	$2\pi\sqrt{gH} / L$	10.1 sec

Where we have used the values H=10 m and L = 100 m.

The patters of surface height for the first two modes look like:



2.ii. To do the scaling we also have to find the velocity solution. We did this for the progressive wave in class, and use the same method here, starting from the x-momentum equation:

$$u_t = -g\eta_x \text{, so...}$$

$$u = (-g)\int \eta_x \, dt = (-g)\int\left[(-k\eta_0)\sin(kx)\cos(\omega t)\right]dt$$
$$= \frac{gk\eta_0}{\omega}\sin(kx)\sin(\omega t) = \frac{\eta_0}{H}\sqrt{gH}\sin(kx)\sin(\omega t)$$

Note that in the last step we have made use of the relation for the phase speed.

Now from our solutions and derivatives thereof it is easy to figure out that the scales of various terms are

$$[\eta] = \eta_0, [\eta_t] = \omega\eta_0, [\eta_x] = k\eta_0$$
$$[u] = \frac{\eta_0}{H}\sqrt{gH}, [u_t] = \omega\frac{\eta_0}{H}\sqrt{gH}, [u_x] = k\frac{\eta_0}{H}\sqrt{gH}$$

and thus the ratios of neglected terms to retained terms are given by

$$\text{MASS:} \quad \frac{[\eta u_x]}{[\eta_t]} = \frac{[\eta u_x]}{[\eta_t]} = \frac{\eta_0}{H} = 0.1$$
$$\text{X-MOM:} \quad \frac{[u u_x]}{[u_t]} = \frac{\eta_0}{H} = 0.1$$

so we were reasonably justified in our approximations. Interestingly these answers are the same for both of our two solution modes.

2.iii. To find the horizontal position of a fluid parcel, call it x^P, just integrate the x-velocity solution in time:

$$x^P = \int u \, dt = \frac{-\eta_0}{H}\frac{\sqrt{gH}}{\omega}\sin(kx)\cos(\omega t)$$

and thus the amplitude is given by $\frac{\eta_0}{\pi}\frac{L}{H}$ for the lowest frequency solution, and $\frac{\eta_0}{2\pi}\frac{L}{H}$ for the next higher frequency. For our lowest mode the amplitude of horizontal motion would be (at its greatest) about 3 m. It is also easy to show that the ratio of this amplitude to the length scale of the wave, k^{-1}, is given by

$$\frac{\frac{\eta_0}{H}\frac{\sqrt{gH}}{\omega}}{k^{-1}} = \frac{\eta_0}{H}\frac{\sqrt{gH}}{\omega / k} = \frac{\eta_0}{H}\frac{\sqrt{gH}}{\sqrt{gH}} = \frac{\eta_0}{H} \text{ for both frequencies.}$$

APPENDIX: MATLAB code to make the figure for Problem 1

```
% ps3_1.m  10/27/2009  Parker MacCready
%
% this plots the various properties for the flow field in
% problem 1

clear
close all
% set some stuff to make nicer figures
fs1 = 14; % fontsize
set(0,'defaultaxesfontsize',fs1);
set(0,'defaulttextfontsize',fs1);
set(0,'defaultaxesfontweight','bold');
set(0,'defaulttextfontweight','bold');
set(0,'defaultaxesfontname','Ariel');

% create the axes
L = 2; % domain limit [m]
x = linspace(-L,L,20); % x-axis [m]
y=x; % y-axis [m]
[X,Y] = meshgrid(x,y);
% create the velocity field
A = 1; % [s-1]
U = A*X; V = -A*Y; % [m s-1]

% Start Plotting
figure

% First add pressure contours
rho = 1.2; % density of air [kg m-3]
P = -0.5*rho*A^2*(X.^2 + Y.^2); % pressure [Pa] to within a constant
[cc,hh] = contour(X,Y,P,[-5:.5:0],'-b');
clabel(cc,hh,'color','b');
hold on

% next add velocity vectors
fact = .1; % this just scales things to they fit
quiver(X,Y,fact*U,fact*V,0,'-k');
% NOTE the 5th argument "0" gets rid of auto-scaling
%
% add a scale arrow
quiver(-.3,.1,fact*1,fact*0,0,'k');
text(-.3,.2,'1 m s^{-1}')

% add two acceleration vectors
afact = 0.5;
% for a point where y=x
C = 1; % constant to define a streamline [m2]
xx = 1; % x-position on that streamline
yy = C/xx; % y-position on that streamline
ax = A^2*xx; % x-direction acceleration at that point [m s-2]
ay = A^2*yy; % y-direction acceleration at that point [m s-2]
quiver(xx,yy,afact*ax,afact*ay,0,'-m');
% same for a different streamline and a point where y=x/10
C = 0.1;
xx = 1;
yy = C/xx;
ax = A^2*xx;
ay = A^2*yy;
quiver(xx,yy,afact*ax,afact*ay,0,'-m');
%
% add a scale arrow
quiver(-.3,-.1,afact*1,afact*0,0,'m');
```

```
text(-.3,-.2,'1 m s^{-2}','color','m')


% plot a few streamlines
xs = linspace(.0001,L,100);
for C = [0.1 1];
    ys = C./xs;
    plot(xs,ys,'-r','linewidth',2)
end

% add labels and fix scaling
axis equal
grid on
axis(L*[-1 1 -1 1]);
xlabel('X'); ylabel('Y')
% fancy multi-color title
title(['\color{black}Velocity \color{blue}Pressure [Pa] ', ...
    '\color{red}Streamlines \color{magenta}Acceleration'])

% and print a copy to a jpeg
print -djpeg stream.jpg
```