FLUIDS 2009

Midterm Exam Solutions 11/6/2009

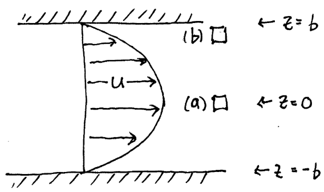
1.i. The velocity is given by $\left(u,v\right)=\left(Ax,-Ay\right)$. The viscous terms in the momentum equations are given by $\nu\nabla^2 u=\nu\left(u_{xx}+u_{yy}\right)=0$ and $\nu\nabla^2 v=\nu\left(v_{xx}+v_{yy}\right)=0$. Hence there is no viscous contribution to fluid acceleration. Physically, there are still viscous stresses, but they have zero divergence.

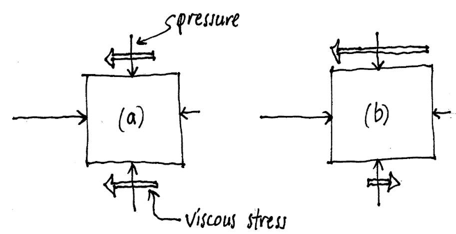
1.ii. Recall that $\frac{p}{\rho}=-\frac{1}{2}A^2\left(x^2+y^2\right)$ (plus some arbitrary, additive constant) from the previous problem set. The Bernoulli function is given by
$\frac{1}{2}\left(u^2+v^2\right)+\frac{p}{\rho}=\frac{1}{2}A^2\left(x^2+y^2\right)-\frac{1}{2}A^2\left(x^2+y^2\right)=0$, which is clearly a constant. Since the flow is 2D there is no need to consider the *gz* term in the Bernoulli function, or it can be absorbed into the constant added to the pressure. In any case the result indicates that the Bernoulli function is constant everywhere, and not just along a streamline.

1.iii. The fact that the Bernoulli function is constant everywhere is consistent with the assumptions used in the derivation because:

- the flow is steady
- the density is constant
- the flow is inviscid - note that this is not strictly true, but for this particular flow we showed in 1.i that viscosity has no influence on the momentum equations, and hence it can have no influence on the integral of the momentum equations along any path
- the flow has zero vorticity: note that the only non-zero component of the vorticity vector is $\hat{k}\left(v_x-u_y\right)=0$

2.i. For parcels at locations (a) and (b) the forces acting on the parcel are just due to pressure and viscous stresses. In the sketch I have assumed no gravity, but if there were gravity then the bottom pressure arrow would be a bit bigger than the top pressure arrow.





2.ii. The x-momentum equation is $\frac{Du}{Dt} = -\frac{1}{\rho} p_x + \nu \nabla^2 u$ which for the simple case of Poiseuille flow reduces to $uu_x = -p_x/\rho + \nu u_{zz}$. This may also be written as $\frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \frac{p}{\rho}\right) = \nu u_{zz}$. Integrating this along a streamline (at constant z), gives: $\left(\frac{1}{2}u^2 + \frac{p}{\rho}\right)\Big|_{x_A}^{x_B} = \int_{x_A}^{x_B} \nu u_{zz}\, dx$. And note that from the flow solution $\nu u_{zz} = p_x/\rho$ which is a negative constant. Hence the flow suffers a decrease in Bernoulli function (or "head") proportional to the distance travelled. This should also be apparent from the physical situation in which a fluid parcel is always moving down the pressure gradient but maintaining the same velocity (in the absence of viscous forces it would have to accelerate).

2.iii. The dissipation rate is given by $\varepsilon = \nu\left(u_z\right)^2 = \frac{p_x^2 z^2}{\rho\mu}$ and so the maximum value occurs adjacent to the top or bottom plates, at $z = \pm b$, hence $\varepsilon_{MAX} = \frac{p_x^2 b^2}{\rho\mu}$. We want to write this in terms of the maximum velocity in the flow, which is given by $u_{MAX} = u\left(z=0\right) = -\frac{1}{2}\frac{p_x b^2}{\mu}$. Thus we may write the maximum dissipation as

$$\varepsilon_{MAX} = 4\nu \frac{\left(u_{MAX}\right)^2}{b^2} \cong 4 \times 10^{-2}\ \frac{\mathrm{W}}{\mathrm{kg}}$$

where we have used the kinematic viscosity $\nu \cong 10^{-6}\ \mathrm{m}^2\ \mathrm{s}^{-1}$ which is typical for water. Then the time required for this to heat water by 1 degree K is given by

$$\frac{T}{1\text{ degree K}} = \frac{C_p}{\varepsilon_{MAX}} = \frac{4182\,\dfrac{\text{J}}{\text{kg K}}}{4\times10^{-2}\,\dfrac{\text{J}}{\text{kg s}}} \simeq \frac{10^5\text{s}}{1\text{ degree K}}$$

which is about a day (not a great way to boil water!).

3. In the original frame of reference the progressive shallow water wave surface elevation and velocity are given by

$$\begin{aligned}\eta &= \eta_0\cos(kx-\omega t)\\ u &= u_0\cos(kx-\omega t)\end{aligned}$$

where $\omega/k = c = \sqrt{gH}$ is the phase speed, and $u_0 = c(\eta_0/H)$ is the magnitude of the velocity [a lot smaller than the phase speed].

If observe this same wave from a frame of reference moving in the positive x-direction at speed c then the fields are

$$\begin{aligned}\eta' &= \eta_0\cos(kx')\\ u' &= u_0\cos(kx') - c\end{aligned} \qquad (*)$$

where we have used a prime to indicate that these are properties measured in the new frame of reference.

The Bernoulli function at the free surface is given by

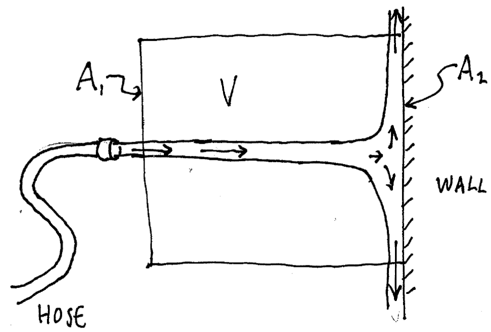
$$\frac{1}{2}u'^2 + \frac{p}{\rho} + gz = \frac{1}{2}u'^2 + \frac{p_{ATM}}{\rho} + g\eta'$$

Is this a constant? Plugging in the expressions (*) above we find

$$\begin{aligned}\frac{1}{2}u'^2 + \frac{p_{ATM}}{\rho} + g\eta' &= \cancel{\frac{1}{2}u_0^2\cos^2(kx')} - u_0c\cos(kx') + \frac{1}{2}c^2 + \frac{p_{ATM}}{\rho} + g\eta_0\cos(kx')\\ &\cong \cancel{-u_0c\cos(kx')} + \frac{1}{2}c^2 + \frac{p_{ATM}}{\rho} + \cancel{g\eta_0\cos(kx')}\\ &= \frac{1}{2}c^2 + \frac{p_{ATM}}{\rho} = \text{constant}\end{aligned}$$

So it is constant, but only if we neglect the term with amplitude $u_0^2/2$. We may do so because it is small (by the factor $\eta_0/H \ll 1$) compared to the term u_0c.

4. Consider a volume that encompasses the flow just after it leaves the hose and extends to the wall. The area of the volume on the wall is big enough so that the flow has completely been turned to be parallel to the wall.



The volume-integrated equation for x-momentum derived in class is:

$$\frac{d}{dt}\int_V \rho u\, dV = -\int_A (\rho u)\mathbf{u}\cdot\hat{\mathbf{n}}\, dA - \int_A p\hat{\mathbf{i}}\cdot\hat{\mathbf{n}}\, dA + \mu\int_A \nabla u\cdot\hat{\mathbf{n}}\, dA$$

and we can evaluate each term to figure out the force on the wall.

Some observations:

The flow is steady, so we can ignore the d/dt term on the left side.

Since the wall is vertical, and we are just concerned with x-momentum we can ignore gravity.

As drawn in the sketch, the water leaving the volume after hitting the wall has $u = 0$, and the only non-zero term due to advection of momentum will be

$$-\int_A (\rho u)\mathbf{u}\cdot\hat{\mathbf{n}}\, dA = \int_{A_1} \rho u^2\, dA = a\rho(u_1)^2$$

where a is the cross-sectional area of the stream as is leaves the hose, with velocity u_1.

The pressure term looks like:

$$\begin{aligned}
-\int_A p\hat{\mathbf{i}}\cdot\hat{\mathbf{n}}\, dA &= \int_{A_1} p\, dA - \int_{A_2} p\, dA \\
&= \cancel{p_{ATM}A_1} + p'a - \cancel{p_{ATM}A_2} - \int_{A_2} p'\, dA \\
&= -\text{extra force on wall due to pressure} \\
&\equiv -F_P
\end{aligned}$$

where we have defined p' as the anomaly of the pressure in the fluid away from atmospheric pressure (so $p = p_{ATM} + p'$). Right where the stream leaves the hose we can assume that $p' = 0$ because the flow is basically steady and unidirectional, and so parcels are not accelerating. Hence we can ignore the $p'a$ term that arose from the integral over A_1. Where the water hits the wall the fluid parcels are decelerating, and this leads to a positive pressure anomaly on the wall. From the point of view of the volume integral this is a force pushing the fluid to the left, and so for the wall it is a force pushing to the right, which we have called F_P. The total force due to pressure on the wall includes a term due to atmospheric pressure, which you could add in for completeness, but it would be irrelevant to the ability of the stream of water to push the wall over.

The viscous terms is only non-zero on the wall side of the volume, and is given by

$$\begin{aligned}\mu \int_{A_2} \nabla u \cdot \hat{\mathbf{n}}\, dA &= \mu \int_{A_2} \frac{\partial u}{dx} dA \\ &= -\text{force on wall due to vicosity} \\ &\equiv -F_V\end{aligned}$$

where we have defined the force on the wall due to viscosity as F_V. The physical interpretation of this term is a little tricky. You can think of it as viscous transfer of momentum when the fluid is very close to the wall. To estimate how big it is compared to the force due to pressure you would have to know more about the details of the flow right where it hits the wall. I expect that it is negligible compared to F_P, but we can retain it for completeness. I would not be unhappy if you assumed the flow was inviscid and just ignored this term.

Now we can figure out the total force on the wall due to the flow, which is given by

$$\begin{aligned}0 &= a\rho\left(u_1\right)^2 - \int_{A_2} p\, dA + \mu \int_{A_2} u_x\, dA \\ &= a\rho\left(u_1\right)^2 - F_P - F_V \\ &= a\rho\left(u_1\right)^2 - F\end{aligned}$$

where we have defined the total extra force on the wall as $F = F_P + F_V$. Thus the answer is given by

$$\begin{aligned}F &= a\rho\left(u_1\right)^2 \\ &= \left(4\times10^{-4}\ \text{m}^2\right)\left(10^3\ \text{kg m}^{-3}\right)\left(4\ \text{m}^2\ \text{s}^{-2}\right) \\ &= 1.6\ \text{N}\end{aligned}$$

The most interesting thing about this whole calculation is that we were able to do it without knowing the details of the pressure distribution on the wall. This is an example of the power of the volume integral method!

5. The total z-direction force balance is a combination of buoyancy pushing up (due to the air displaced by the balloon, $gV\rho_{AIR}$) and gravity pushing down (due to the mass m of the boy and the mass of helium gas in the balloon, $-mg - gV\rho_H$).

$$0 = -mg + gV\left(\rho_{AIR} - \rho_H\right)$$
$$\Rightarrow V = \frac{m}{\left(\rho_{AIR} - \rho_H\right)} = \frac{20\text{ kg}}{\left(1.2\text{ kg m}^{-3} - 0.2\text{ kg m}^{-3}\right)} = 20\text{ m}^3$$

so I conclude that the news story was plausible, but still seemed fishy.

6. Looking forward to your answers!