FLUIDS 2009

Final Exam 12/7/2009

Due Monday 12/14/2009 11:30 AM in the box outside Ocean Sciences Building 313

Open book (any book), open notes, but work on your own

1.a Derive the functional form for pressure as a function of radius for a Rankine vortex. Assume that the azimuthal velocity is *U* at radius *R*, the edge of the core of solid body rotation. Prove that the total pressure drop is independent of *R*, and only depends on *U* and the fluid density (assume density is constant). What fraction of the total pressure drop is achieved at the edge of the core of solid body rotation? [10]

1.b. For a hurricane approximated by such a functional form, what is the total pressure drop in the center of the storm if the azimuthal velocity is $u_\theta(R) = U = 90 \text{ m s}^{-1}$ at the radius of the core of solid body rotation? What fraction of standard atmospheric pressure at sea level is this? You can use $p_{ATM} \cong 10^5 \text{ Pa}$. [5]

1.c. Now consider the case where this is a vortex in water. Translate your result from 1.a into an expression for the free surface height as a function of radius. Then derive analytical expressions for the volume integrated kinetic and potential energies as functions of *U*, *R*, and the undisturbed water depth *H*. The shallow water form of these quantities will be sufficient, and you may ignore variation of water depth when calculating the expression for net kinetic energy. [10]

2.a. Consider a shallow water wave with friction. The dynamics are approximately given by the equations:

$$\text{X MOM} \qquad \frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} - Ru$$

$$\text{MASS} \qquad \frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

where *R* is a "Rayleigh friction" (not to be confused with the radius in problem 1!). Formally X MOM is derived as a vertical average of the full equation, averaging through a turbulent bottom boundary layer, and then linearizing the drag term. Derive a single equation from these two for the evolution of free surface height $\eta(x,t)$. [10]

2.b. Find the dispersion relation for these waves. It is helpful to guess a form of the solution $\eta = \eta_0 \operatorname{Re}\{\exp[i(kx - \omega t)]\}$ and allowing the frequency to have real and imaginary parts. [5]

2.c. Are these waves dispersive? [5]

2.d. For waves with period 12.42 hours (the dominant tidal period on our planet) and $R = 10^{-5}\ \text{s}^{-1}$ what is the wavelength? Use water depth 200 m. [5]

3.a. For 2D potential flow around a cylinder we found that the net drag on the cylinder was zero. Consider instead the case in which the background flow U is a function of time. Using the generalized Bernoulli function find the expression for the pressure on the surface of the cylinder including the unsteady term. [15]

3.b. Integrate the x-component of force on the cylinder due to pressure to find the analytical expression for the net x-direction force per unit length of cylinder. The force should be non-zero for non-zero $\partial U/\partial t$. [10]