Introduction to Fluid Dynamics
Problem Set 3, 10/26/2007, Due at the start of class 11/2/2007

Consider a round disk of radius R, thickness H and mass M falling through water of constant density $\rho_0 = 1000$ kg m^{-3} onto a flat surface. As the disk approaches the surface it pushes water out from under it with radial velocity $u(r,z,t)$, where r is the radial coordinate with zero at the center of the disk. The vertical coordinate, z, is positive upwards, with $z=0$ at the flat surface, and $z=h(t)$ at the lower face of the disk. The velocity of the disk is purely vertical, and is given by $\partial h/\partial t$, a negative number. If the disk were not moving we could calculate the pressure everywhere by the hydrostatic relation, and easily figure out the buoyancy of the disk. However, when the disk is falling, as it gets close to the surface it must push water out to the sides through an ever-narrowing gap. This can give rise to viscous stresses which can raise the pressure under the disk, slowing its descent. This is what we are trying to calculate here. The fundamental equations of mass and momentum conservation in cylindrical coordinates will be convenient to use, and they are given in Appendix B1 of Kundu and Cohen. Note that what they call x in their notation, we are calling z. Also, what they call u_R is what we mean by u, and what they call R we are calling r.

A[5]. We assume the flow is incompressible. For flow in the "gap" between the disk and the flat surface the remaining terms in the mass conservation equation are

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0$$

where w is the vertical velocity. Take the vertical integral of this from 0 to h to derive an expression for $\overline{u}$, the vertically-averaged radial velocity in the gap.

B[10]. Assume that the radial momentum equation can be approximated as a quasi-steady balance between the radial pressure gradient and the vertical viscous stress divergence. By "quasi-steady" we mean that time derivatives do not appear explicitly in the equation, but some of the coefficients may be time-varying. Thus r-mom is:

$$0 = -\frac{\partial p}{\partial r} + \mu\frac{\partial^2 u}{\partial z^2}$$

Derive an expression for $u(z)$ assuming the "no-slip" boundary condition $u(z=0) = u(z=h) = 0$. There will still be variation of u in radius and time, but it will be contained in the coefficients of your expression. You may assume that the gap is thin enough so that p is not a function of z there.

C[5]. Use the results of B to derive another expression for $\overline{u}$.

D[5]. Use the results of A and C to find a solution for $p(r)$ to within a constant of integration. Note that again the time dependence will be contained within the coefficients of your expression.

E[5]. Assume that the full force balance on the disk is given by gravity, buoyancy, and the area integral of the extra pressure under the disk arising from viscous friction. Taking the pressure at the edge of the disk, $z = 0$, and $r = R$ to be p_0 (the hydrostatic pressure at $z = 0$), find the full expression for $p'(r) = p(r) - p_0$ in the gap. Thus we are calling the increase of pressure over its hydrostatic value p'. Again, the time dependence will be contained within the coefficients of your expression, and you may assume that the gap is thin enough to neglect vertical variation of p within it.

F[10]. Considering the whole force balance on the disk as in E, derive an expression for $\partial h/\partial t$. This can be expressed as a function of $h(t)$.

G[5]. What speed is the disk falling at if we take $R = 1\text{ m}$, $H = 5\text{ cm}$, $M = 1000\text{ kg}$, $\mu = 10^{-3}\text{ kg m}^{-1}\text{ s}^{-1}$? Compute your answer for $h = 1\text{ cm}$, $h = 1\text{ mm}$, and $h = 0.1\text{ mm}$.

H[20]. For the case $h = 1\text{ mm}$, look at the full version of r-mom in the Appendix of Kundu and Cohen. Use your solution to calculate the magnitude of some of the terms we neglected. Were we justified?