Introduction to Fluid Dynamics: Final Exam 12/5/2007

Due at Noon 12/12/2007 in the box outside my office (313 OSB)

Open book (KC), open notes, no collaboration

1[total=25]. Assuming that the pressure is hydrostatic (and thus determined by the horizontal gradient of the surface height) …

A[15]. …what is the functional form of the surface height field for a Rankine vortex in water with a free surface?

B[10]. Assume that scales are similar to the "tidal headland eddy" formed by horizontal flow separation of tidal flow past a point in Puget Sound, as shown in the figure. Azimuthal velocity at the edge of the core of solid body rotation is about $u_\theta = 30$ cm s^{-1}, and the core has radius 500 m. What is the maximum deformation of the free surface height at the center of the eddy (make sure you specify the sign!), and what is its value at the edge of the core?

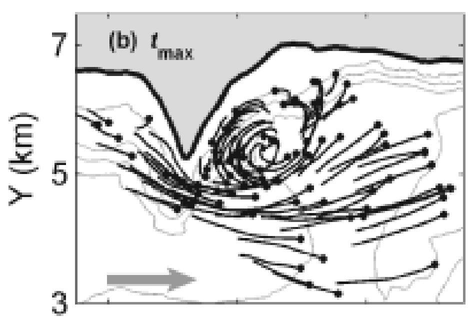

Figure: surface Lagrangian drifter tracks during tidal flow past Three Tree Point in Puget Sound, during maximum flood current. The location of each drifter is shown with a dot, and the last hour of its track is shown with a line. The gray region at the top is land. The Point is about 1 km in scale. A large "tidal headland eddy" with positive vertical vorticity is evident to the right of the Point.

2[25]. Sketch the distributions of velocity, pressure, and density in a sound wave. Discuss the energetics of a fluid parcel as sound waves pass through it. When (in relation to times of peak pressure) is kinetic energy added to the parcel, and when is internal energy added to it? Is the change of internal energy reversible?

3[total=50]. Consider shallow water waves with bottom friction. In the "shallow water" limit (the wavelength, λ, is much greater than H, the undisturbed water depth) the velocity is predominantly horizontal ($u \gg w$) and is approximately independent of depth. Thus $u = u(x,t)$. The governing linearized equations are

X-MOM $\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} - Ru$

MASS $\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0$

where $\eta(x,t)$ is the elevation of the fluid surface away from its rest position (i.e. it equals zero when the water surface is undisturbed). The term $-Ru$ is a simplified representation of the effect of bottom friction, and R is a constant with units of s^{-1}. Formally one would derive an equation like X-MOM above by taking a *vertical average* of the full x-momentum equation.

A[5]. Taking $R = 0$ at first, combine X-MOM and MASS into a single "wave" equation for $\eta(x,t)$. This will have the form $\eta_{tt} - c^2\eta_{xx} = 0$. This is a different way of getting to the shallow water limit of the surface gravity wave solution presented in class.

B[5]. Use the results of A to determine the expression for $\omega(k)$ (the dispersion relation). Hint: guess a solution of the form $\eta = \eta_0 \operatorname{Re}\{\exp[i(kx - \omega t)]\}$. Are these waves dispersive? What is the wavelength if the wave period is 12 hours for water 10 m deep? This is like tides in a river/estuary channel.

C[5]. For $u = u_0 \cos(kx - \omega t)$, find an expression for u_0 in terms of c, η_0, and H, where $c \equiv \omega/k = (gH)^{1/2}$. What is u_0 for $\eta_0 = 1 \text{ m}$ and $H = 10 \text{ m}$? Does this satisfy the linearization assumption $U/c \ll 1$?

D[10]. Repeat A, but now assume non-zero R. The resulting equation will be different from the classical wave equation.

E[10]. Repeat B with non-zero R. Hint: ω will be complex, and you may assume that the real part of the frequency (call this ω') is positive. Describe the behavior of the solution in words or sketches.

F[10]. For waves like those in C, a reasonable value for R would be 10^{-3} s^{-1} (assuming a rough bottom). With this value of the bottom friction, how much is the phase speed $c' = \omega'/k$ different from $(gH)^{1/2}$? Assume that the wavelength is 100 km.

G[5]. How much is the wave amplitude decreased by bottom friction in the time it takes the wave to travel one wavelength?