Perron Spectratopes and the Real Nonnegative Inverse Eigenvalue Problem

UWB Mathematics Society

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Overview

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- Denote by e_i the i^{th} standard basis vector of $\mathbb{R}^n$.
- Let $e := [1 \ \cdots \ 1]^\top \in \mathbb{R}^n$.

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Applications of PFT:

- Continued fractions.
- Internet search engines (e.g., Google Matrix).
- Resource-allocation in wireless networks.
- Probability theory (ergodicity of Markov chains).
- Symbolic dynamics/dynamical systems (subshifts of finite type).
- Economics (e.g., Okishio's theorem, Leontief's input-output model, Walrasian stability of competitive markets).
- Demography (Leslie model).
- Ranking methods (e.g., football teams).
- Low-dimensional topology.
- Statistical mechanics.
- Epidemiology (Kermack-McKendrick threshold).
- Matrix iterative analysis (Stein-Rosenberg theorem).

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  - Karpelevich (1951) gave an implicit, parametric description of Θ_n for every $n \in \mathbb{N}$.

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$$\rho(\sigma) := \max_{i \in \langle n \rangle} |\lambda_i| \in \sigma \qquad (\text{PFT})$$

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- Holtz (2004) showed that if σ is realizable, where $\lambda_1 = \rho(\sigma)$, then the shifted spectrum $\{0, \lambda_1 - \lambda_2, \ldots, \lambda_1 - \lambda_n\}$ satisfies *Newton's inequalities*.

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  - SNIEP is unsolved for $n \geq 5$.
- It is known that RNIEP and SNIEP are equivalent when $n \leq 4$ but distinct otherwise (Johnson, Laffey, & Loewy 1996).

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Definition (Polyhedron & Polytope)

A polyhedron is any set of the form $\mathcal{P}(A, b) := \{x \in \mathbb{R}^n : Ax \leq b\}$, where A is an m-by-n real matrix and $b \in \mathbb{R}^m$. A polyhedral cone is any polyhedron of the from $\mathcal{P}(A, 0)$. A polytope is a bounded polyhedron.

- For $k \in \langle n \rangle$, let P_k be the $(n-1)$-by-n matrix obtained by deleting the k^{th}-row of I and define $\pi_k : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ by $\pi_k(x) = P_k x$.

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- Since $SIS^{-1} = I \geq 0$ for every invertible matrix S, it follows that $\mathcal{P}(S)$ is always nonempty; if $n \geq 2$, then $\mathcal{P}^1(S)$ is always nonempty.

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- Let
$$\mathcal{P}(S) := \{x \in \mathcal{C}(S) : x_1 = 1\}$$
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- The sets $\mathcal{P}\left(S\right)$ and $\mathcal{P}^1(S)$ are polytopes.
- We refer to $\mathcal{P}\left(S\right)$ as the (Perron) spectratope of S and $\mathcal{P}^1(S)$ as the projected (Perron) spectratope of S.

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Proof.

Necessity. If $\operatorname{coni}(e) \subset \mathcal{C}(S)$, then there is a vector $x \neq e$ such that $A := SD_x S^{-1} \geq 0$. Following the PFT, there is an $i \in \langle n \rangle$ such that Se_i and $e_i^\top S^{-1}$ are both nonnegative.
Sufficiency. If $x := Se_i \geq 0$ and $y^\top := e_i^\top S^{-1} \geq 0$, then $SD_{e_i} S^{-1} = xy^\top \geq 0$. Thus, $\operatorname{coni}(e) \subset \mathcal{C}(S)$. □

Theorem

Let

$$S = \begin{bmatrix} s_1^\top \\ \vdots \\ s_n^\top \end{bmatrix}.$$

If $y^\top := e_i^\top S^{-1}$, *then* $y \geq 0$ *if and only if* $e_i \in \operatorname{coni}(s_1, \ldots, s_n)$. *Moreover,* $y > 0$ *iff* $e_i \in \operatorname{int}(\operatorname{coni}(s_1, \ldots, s_n))$.

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Corollary

If

$$S = \begin{bmatrix} s_1^\top \\ \vdots \\ s_n^\top \end{bmatrix} \text{ and } (S^{-1})^\top = \begin{bmatrix} t_1^\top \\ \vdots \\ t_n^\top \end{bmatrix},$$

then S *is a Perron-similarity iff there is an* $i \in \langle n \rangle$ *such that* $e_i \in \operatorname{coni}(s_1, \dots, s_n)$ *and* $e_i \in \operatorname{coni}(t_1, \dots, t_n)$.

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Corollary

The spectratope of the Walsh matrix of order 2^n is the convex hull of its rows.

NIEP = RNIEP = SNIEP: $n = 2$

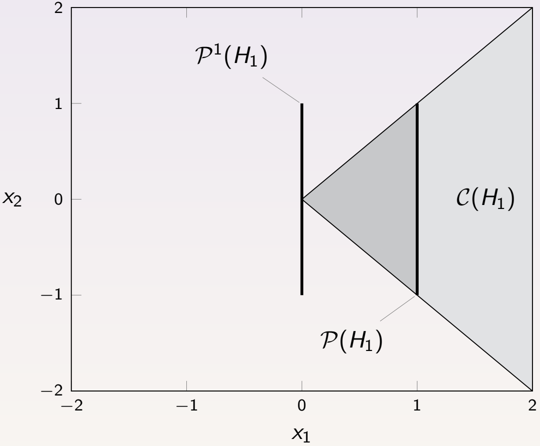

Figure: $H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

RNIEP: $n = 3$

Let

$$S := \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } P := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

For $a \in [0,1]$, let $b := 1 - a$ and

$$S_a := \begin{bmatrix} 1 & 1 & 0 \\ 1 & -a & 1 \\ 1 & -a & -1 \end{bmatrix}.$$

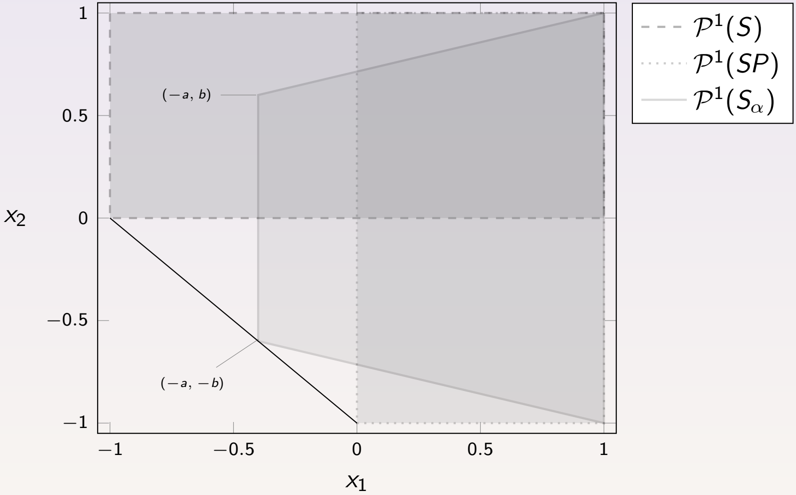

Figure: RNIEP & SNIEP for $n = 3$.

NIEP: $n = 3$

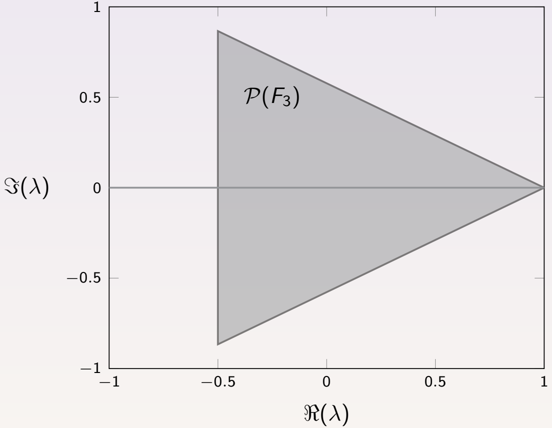

Figure: $F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$.

RNIEP: $n = 4$

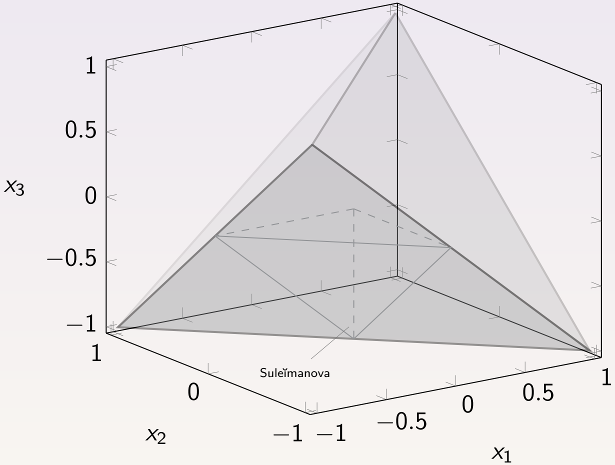

Figure: $\mathcal{P}^1(H_2)$.

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- Egleston et al. (2004) posed the problem of finding a geometric representation in $\mathbb{R}^3$ of all sets of the form $\{1, \lambda, \alpha + \mathrm{i}\omega, \alpha + \mathrm{i}\omega\}$ which are solutions to the NIEP.

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- Benvenuti (2014) solved the problem posed by Egleston et al. using the main result from Torre-Mayo et al (18 pages).

NIEP: $n = 4$

Θ_4

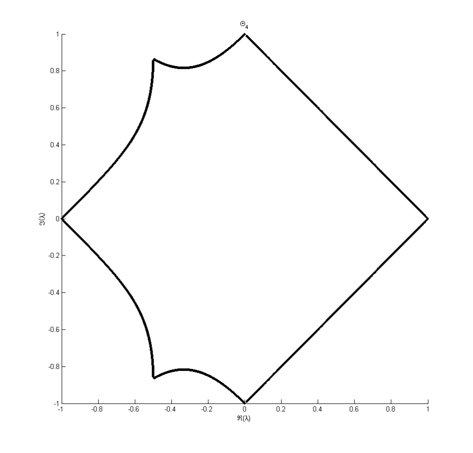
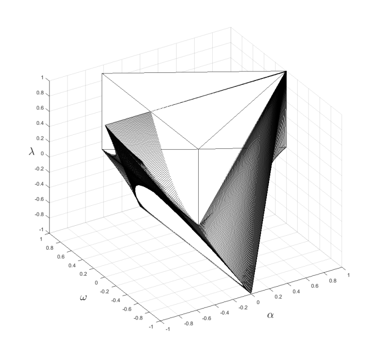
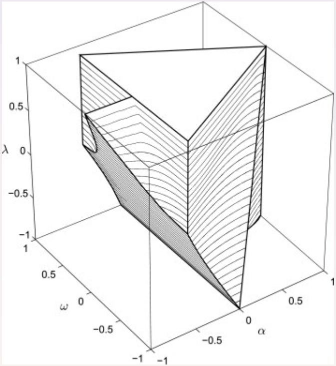


Figure: Θ_4

NIEP: $n = 4$



(a) Via Spectratopes



(b) Benvenuti

Other work

- Generalize previous technique for $n \geq 5$.

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- SNIEP: characterize the spectratopes of *Householder transformations*: $H = I - 2(vv^\top)$, $v^\top v = 1$.

Thank you!