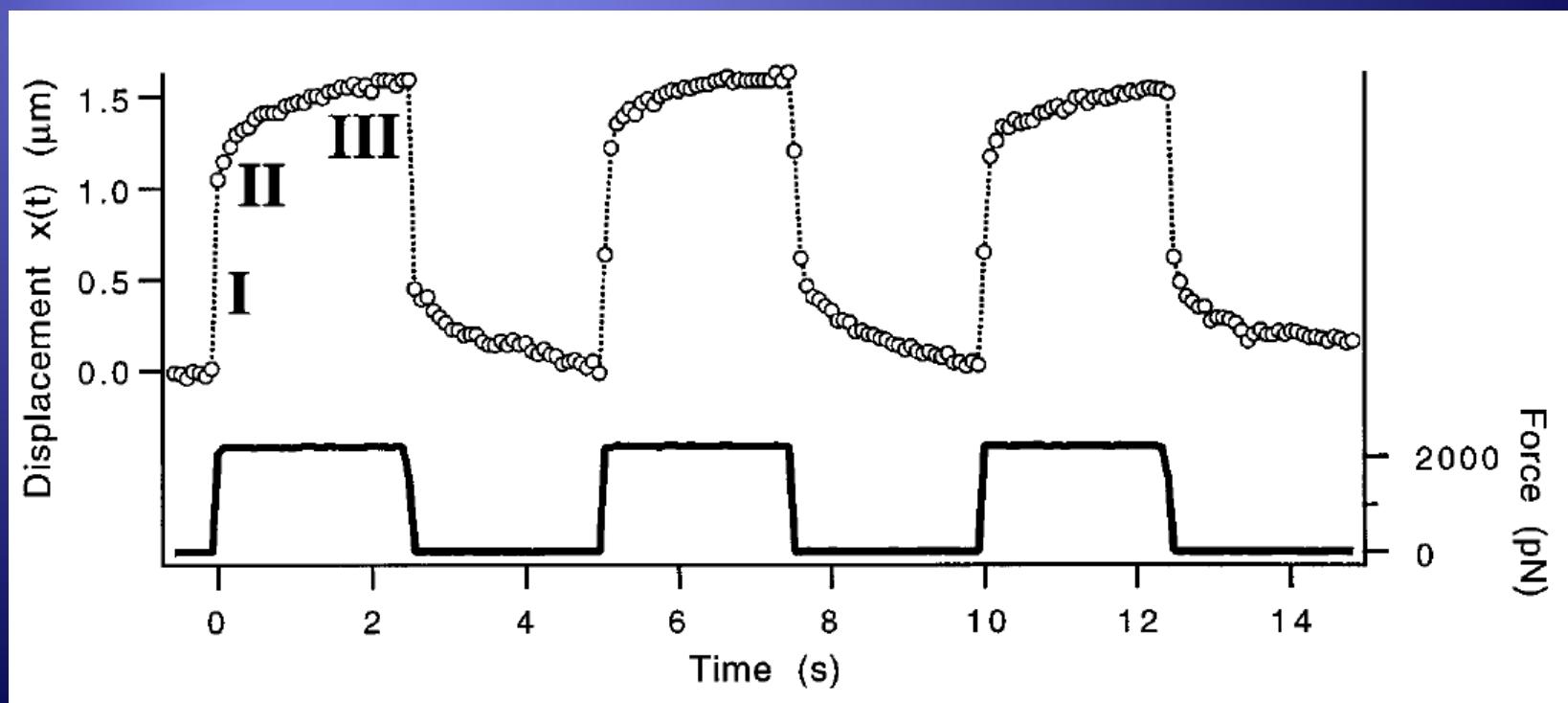


Session 12

# LUMPED PARAMETER MODELS

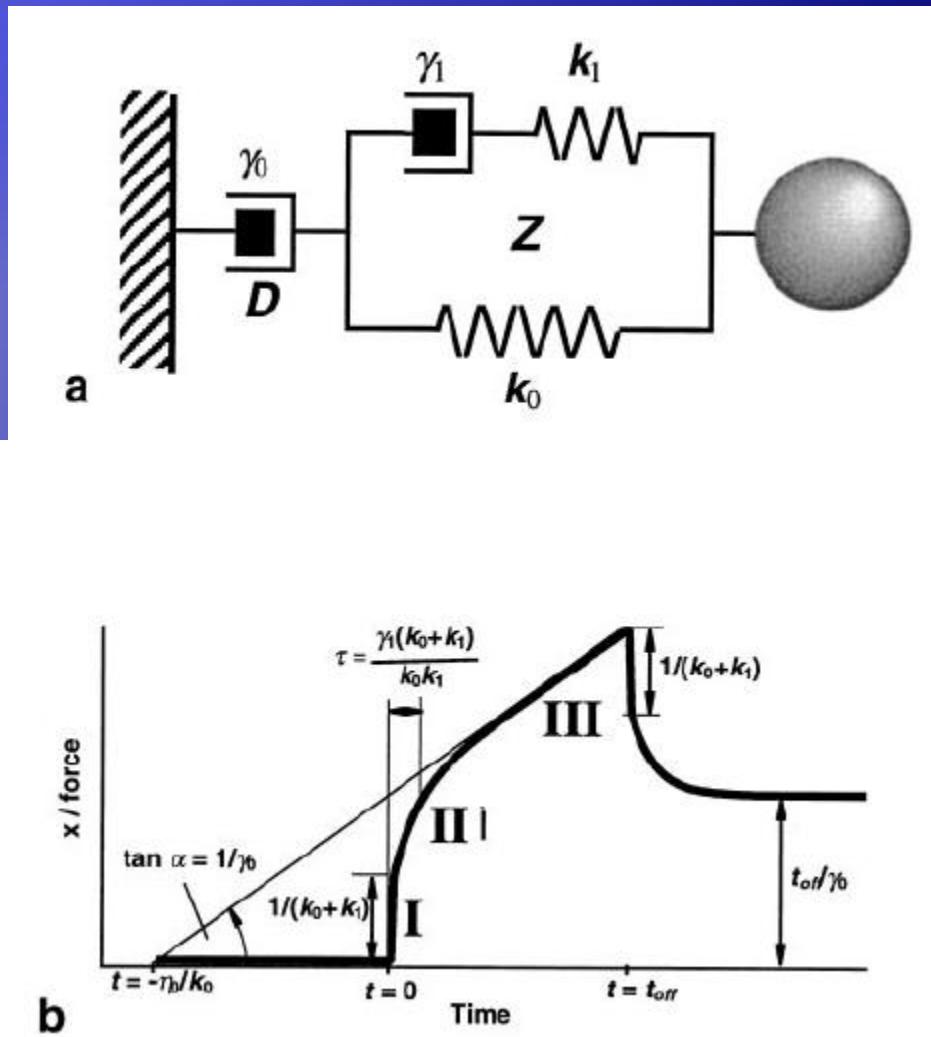
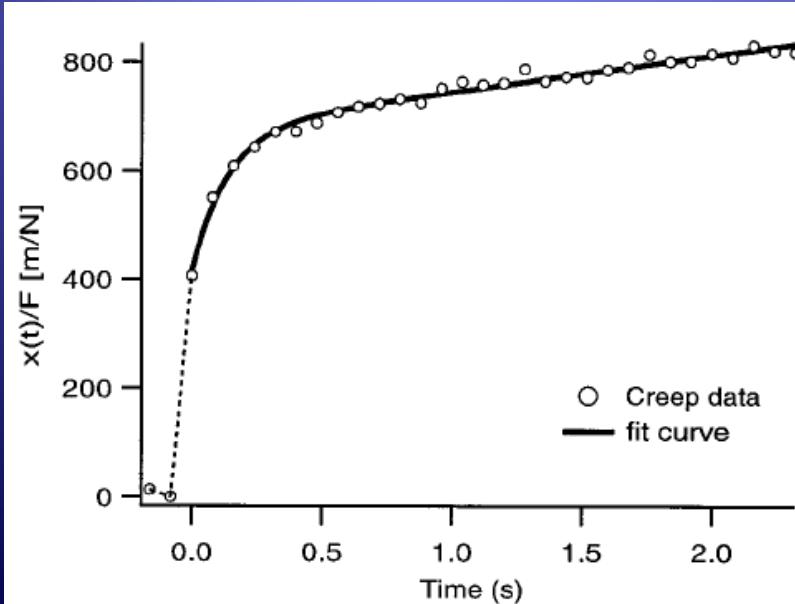
# Recall Bausch et al.

- ◆ Classic Creep Response
  - ◆ Elastic displacement (Regime I)
  - ◆ Relaxation (Regime II)
  - ◆ Steady-state Flow (Regime III)



# The Model

- ◆ Effective elastic modulus  $k = k_1 + k_2$
- ◆ Viscosity  $\gamma_0$
- ◆ Relaxation time  $\tau$



# Springs and Dashpots

- ◆ Force is proportional ( $k$ ) with distance

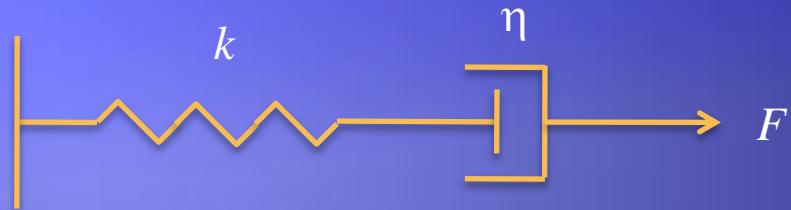


- ◆ Force is proportional ( $\eta$ ) with velocity



# Maxwell Body

- ◆ Spring and dashpot in series



- ◆ Displacement is combined

$$x = x_{sp} + x_{dp}$$

$$\dot{x} = \dot{x}_{sp} + \dot{x}_{dp}$$

$$\ddot{x} = \frac{\dot{F}}{k} + \frac{F}{\eta}$$

# Maxwell Body

- ◆ Consider applying a step force

$$F(t > 0) = F$$

$$\dot{F} = 0$$

- ◆ Differential equation becomes

$$\dot{x} = \frac{\dot{F}}{k} + \frac{F}{\eta}$$

$$\dot{x} = 0 + \frac{F}{\eta}$$

$$x = \frac{F}{\eta} t + C$$

# Maxwell Body

- Initially, dashpot has zero velocity so displacement is only from spring

$$x(0^+) = \frac{F}{k} = 0^+ + C$$

- Thus,

$$x = \frac{F}{\eta} t + C$$

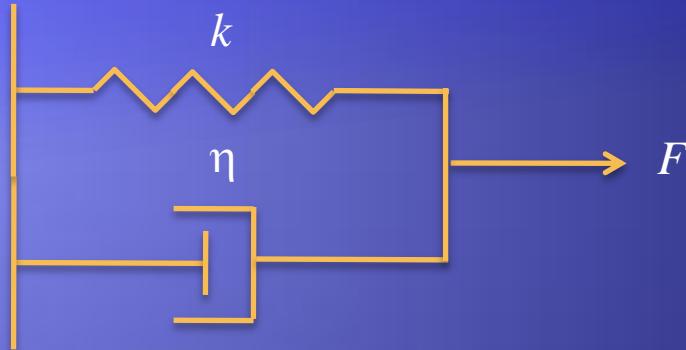
$$x = \frac{F}{\eta} t + \frac{F}{k}$$

$$x = \left( \frac{1}{\eta} t + \frac{1}{k} \right) F$$



# Voigt Body

- ◆ Spring and dashpot in parallel



- ◆ Sum of forces at cross-bar

$$F = F_{sp} + F_{dp}$$

- ◆ Since displacements are the same

$$F = kx + \eta\dot{x}$$

# Voigt Body

- ◆ Have 1<sup>st</sup> ODE

$$\dot{x} + \frac{k}{\eta} x = \frac{F}{\eta}, \quad x(0) = 0$$

- ◆ Use integration factor

$$\mu = \exp\left(\int \frac{k}{\eta} dt\right) = \exp\left(kt / \eta\right)$$

$$\exp\left(kt / \eta\right) \left( \dot{x} + \frac{k}{\eta} x \right) = \frac{F}{\eta} \exp\left(kt / \eta\right)$$

$$\frac{d}{dt} (\exp(kt / \eta) x) = \frac{F}{\eta} \exp(kt / \eta)$$

$$\exp(kt / \eta) x = \frac{F}{k} \exp(kt / \eta) + C$$

# Voigt Body

- ◆ Use initial condition

$$\exp(kt/\eta)x = \frac{F}{k} \exp(kt/\eta) + C$$
$$\Rightarrow 0 = \frac{F}{k} + C$$

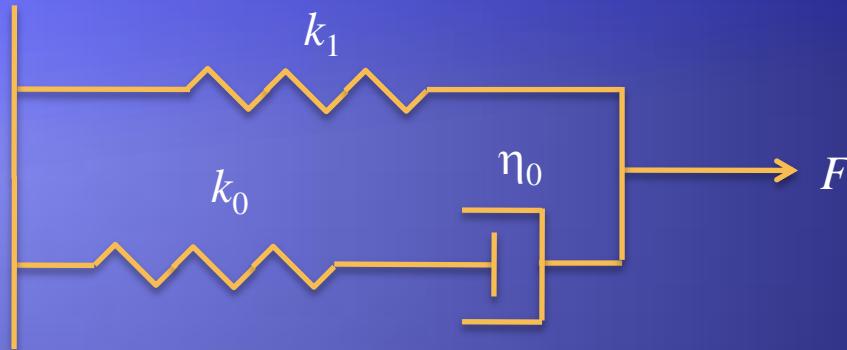
- ◆ Thus

$$x = \frac{F}{k} (1 - \exp(-kt/\eta))$$



# Kelvin Body

- ◆ (Standard Linear Solid)



- ◆ Force is combined

$$F = F_{k1} + F_{maxwell}$$

$$\Rightarrow F_{maxwell} = F - k_1 x$$

# Kelvin Body

- ◆ Recall Maxwell body equation

$$\dot{x} = \frac{\dot{F}_{maxwell}}{k_0} + \frac{F_{maxwell}}{\eta_0}$$

- ◆ Using previous equation

$$\dot{x} = \frac{(\dot{F} - k_1 \dot{x})}{k_0} + \frac{(F - k_1 x)}{\eta_0}$$

- ◆ Rearranging

$$k_1 \left( x + \frac{\eta_0}{k_1} \left( 1 + \frac{k_1}{k_0} \right) \dot{x} \right) = F + \frac{\eta_0}{k_0} \dot{F}$$

# Kelvin Body

- ◆ Substitute  $\tau$  for 1<sup>st</sup> ODE

$$\tau = \frac{\eta_0}{k_1} \left( 1 + \frac{k_1}{k_0} \right)$$

$$k_1(x + \tau \dot{x}) = F + \frac{\eta_0}{k_0} \dot{F}$$

- ◆ For “step” force, simplifies to

$$F(t > 0) = F$$

$$\dot{F} = 0$$

$$k_1(x + \tau \dot{x}) = F + 0$$

$$\dot{x} + \frac{1}{\tau} x = \frac{F}{k_1 \tau}$$

# Kelvin Body

- ◆ Use integration factor

$$\dot{x} + \frac{1}{\tau}x = \frac{F}{k_1\tau}$$

$$\mu = \exp(t/\tau)$$

$$\exp(t/\tau) \left( \dot{x} + \frac{1}{\tau}x \right) = \exp(t/\tau) \frac{F}{k_1\tau}$$

$$\exp(t/\tau)x = \int \exp(t/\tau) \frac{F}{k_1\tau} dt$$

$$\exp(t/\tau)x = \exp(t/\tau) \frac{F}{k_1} + C$$

# Kelvin Body

- ♦ Initially (again), dashpot has zero velocity so displacement is only from **springs**

$$F = k_0 x(0^+) + k_1 x(0^+)$$

$$x(0^+) = \frac{F}{k_0 + k_1}$$

- ♦ Thus

$$\exp(t/\tau)x = \exp(t/\tau)\frac{F}{k_1} + C$$

$$\frac{F}{k_0 + k_1} = \frac{F}{k_1} + C \Rightarrow C = F\left(\frac{1}{k_0 + k_1} - \frac{1}{k_1}\right)$$

# Kelvin Body

- ◆ Simplifying

$$x = \frac{F}{k_1} + F \left( \frac{1}{k_0 + k_1} - \frac{1}{k_1} \right) \exp(-t / \tau)$$

$$x = \frac{F}{k_1} + \frac{F}{k_1} \left( \frac{k_1}{k_0 + k_1} - 1 \right) \exp(-t / \tau)$$

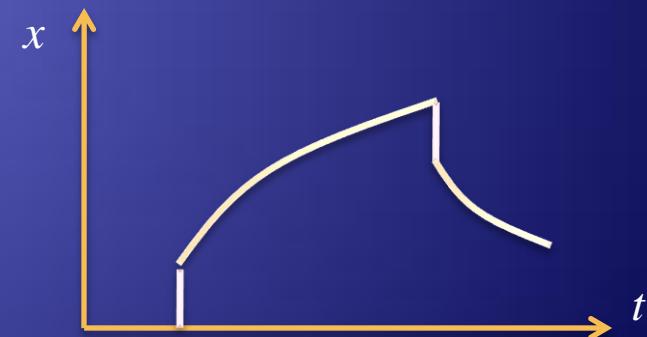
$$x = \frac{F}{k_1} \left( 1 + \left( \frac{k_1}{k_0 + k_1} - \frac{k_0 + k_1}{k_0 + k_1} \right) \exp(-t / \tau) \right)$$

$$x = \frac{F}{k_1} \left( 1 - \left( \frac{k_0}{k_0 + k_1} \right) \exp(-t / \tau) \right)$$

# Kelvin Body

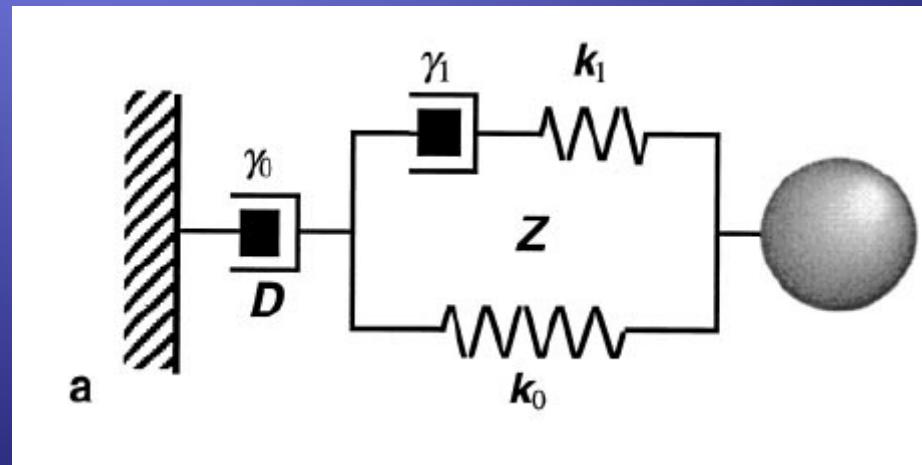
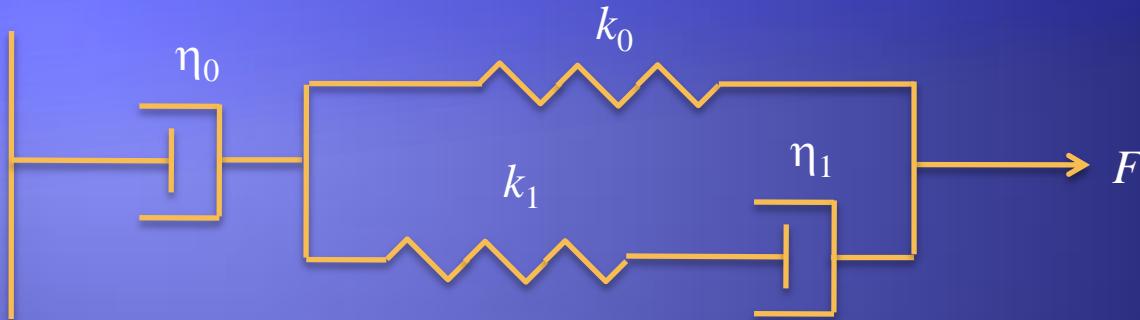
- ◆ Graphically

$$x = \frac{F}{k_1} \left( 1 - \left( \frac{k_0}{k_0 + k_1} \right) \exp(-t / \tau) \right)$$



# Bausch et al.

- ◆ Two spring, two dashpots



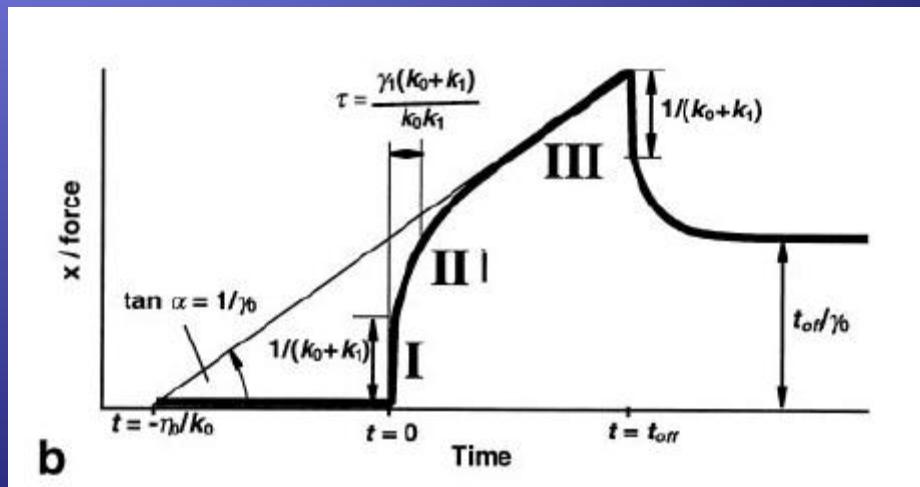
# Bausch et al.

- ◆ By superposition

$$\frac{x}{F} = \frac{1}{k_0} \left( 1 - \left( \frac{k_1}{k_0 + k_1} \right) \exp(-t/\tau) \right) + \frac{t}{\eta_0}$$

Note: difference in subscripts from Kelvin body

- ◆ Graphically



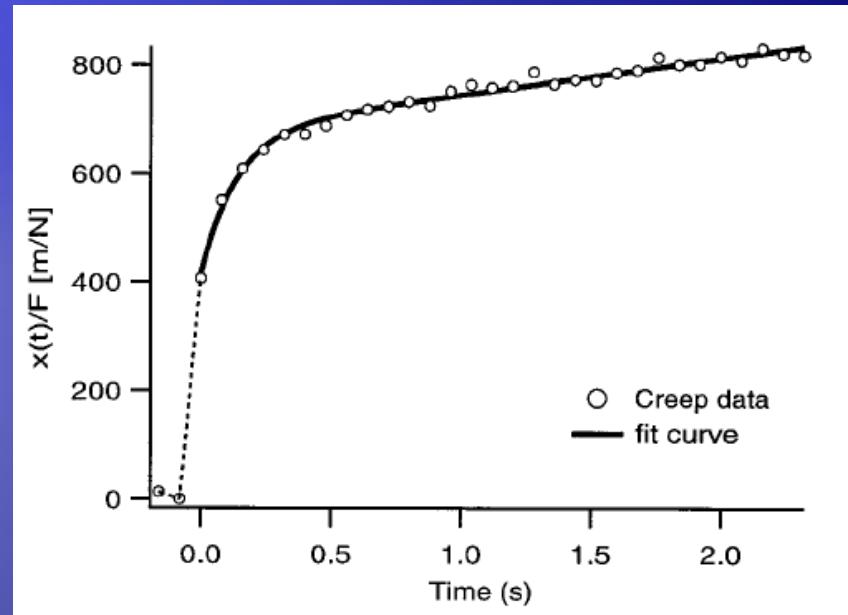
# Regression Analysis

- ◆ Least Squares Fit

$$y = Ax + B$$

$$\Rightarrow A = \frac{\sum x^2 \sum y - \sum x \sum xy}{N \sum x^2 - (\sum x)^2}$$

$$\Rightarrow B = \frac{N \sum xy - \sum x \sum y}{N \sum x^2 - (\sum x)^2}$$



- ◆ More complex fitting requires statistical package

$$\frac{x}{F} = \frac{1}{k_0} \left( 1 - \left( \frac{k_1}{k_0 + k_1} \right) \exp \left( -\frac{\eta_1 (k_0 + k_1)}{k_0 k_1} t \right) \right) + \frac{t}{\eta_0}$$