

## ANALYSIS AND MODELING OF CELL MECHANICS

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### Homework #5 (due 2/27/13)

The learning objective of this homework is to understand the rheological properties of a cell as it relates to the different viscoelastic models, complex modulus, storage modulus, and loss modulus.

Recall Euler's Formula:

$$e^{ix} = \cos x + i \sin x$$

where  $i$  is the imaginary unit and  $x$  is given in radians. Consider the simple harmonic motion of a microbead on the surface of a cell under external force. The displacement will be of the form

$$x = A \cos(\omega t) \tag{1}$$

where  $A$  is the amplitude and  $\omega$  is the frequency in radians per second. Harmonic motion can be due a magnetic bead moving laterally moves to a magnetic twisting field or a latex bead moving under an oscillating beam of an optical trap.

We can write eq. 1 in polar form,

$$x = A e^{i\omega t} . \tag{2}$$

The derivative with respect to time is

$$\dot{x} = \frac{d}{dt} (A e^{i\omega t}) = A i \omega e^{i\omega t} \tag{3}$$

Notice that by combining eq. 2 and 3,

$$\dot{x} = i \omega x \tag{4}$$

This is a useful expression for derivatives that we will apply to the lumped parameter viscoelastic models. Recall the differential equation for a Maxwell body

$$\dot{x} = \frac{\dot{F}}{k} + \frac{F}{\eta} \tag{5}$$

This can be written in polar form of a complex number

$$iwx = \frac{iwF}{k} + \frac{F}{\eta} \quad (6)$$

We will now introduce the complex modulus  $G^*$  by noting that it is can be regarded as the ratio of force to displacement.\*

$$F = G^*x \quad (7)$$

With this, we can write eq. 6 as

$$iwx = \frac{iwG^*x}{k} + \frac{G^*x}{\eta} \quad (8)$$

$$iw = \left( \frac{iw}{k} + \frac{1}{\eta} \right) G^* \quad (9)$$

This expression can be rearranged into the following

$$G^* = \frac{w^2k\eta^2}{w^2\eta^2 + k^2} + \frac{wk^2\eta}{w^2\eta^2 + k^2}i \quad (10)$$

The real part of the complex modulus is the storage modulus and the imaginary part is the loss modulus ( $G^* = G' + iG''$ ). We can easily read off the two distinct terms in eq. 10 to obtain the rheological properties for the model

$$G' = \frac{w^2k\eta^2}{w^2\eta^2 + k^2} \quad (11)$$

$$G'' = \frac{wk^2\eta}{w^2\eta^2 + k^2} \quad (12)$$

Using a similar approach, what is the storage modulus and loss modulus for a) Voigt and b) Kelvin bodies?†

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\* A more formal definition for the complex modulus is the ratio of shear stress to shear strain under oscillatory conditions.

† For a Kelvin viscoelastic model, reported model parameters from micropipette aspiration data have a range from  $k_1 = 5-360$  Pa,  $k_0 = 22-2300$  Pa,  $\eta_0 = 6-4,300$  Pa·s. Lim, C.T., et al. (2006) "Mechanical models for living cells - a review", J. Biomechanics, 39:195-216.

Reference:

Y.C. Fung, (1993) "Biomechanics: Mechanical Properties of Living Tissues", Springer. (Chapter 2.12)